Magnetohydrodynamic Stability and Thermonuclear Containment

A Collection of Reprints
With an Introductory Review by

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An Introduction | by A. JEFFREY AND T. TANIUTI

During the recent decade of progress in the confinement of high temperature plasmas, instabilities of various different types have been encountered. As a general rule these instabilities may be classified into one of two groups: the macroscopic instabilities and the microscopic instabilities. The former are concerned with phenomena at low frequency and long wavelength in which the plasma may be assumed to be neutral in charge. In the latter, the frequency of the micro-instabilities is not necessarily low, the collective electric field often plays an essential role, and the velocity distribution of the plasma is not necessarily Maxwellian. Consequently it then becomes necessary to work in terms of kinetic theory and to solve the coupled Vlasov and Maxwell equations.

The present state of knowledge is such that it is not always clear which observable effects may be directly attributed to microscopic instabilities, and no discussion of such matters will be attempted in this introduction. In the case of macro-instabilities, which will form the object of this review and collection of papers, the phenomena of interest vary so slowly that the displacement current may be neglected, and the Larmor radius of the ions as well as of the electrons may be assumed to be zero in the zeroth approximation (i.e., m/e may be set equal to zero). Hence the discussion of macroscopic instabilities is usually based on magnetohydrodynamics or, for a collisionless plasma, on the drift approximation.

For plasma motion in a direction transverse to the magnetic field, the drift approximation may equivalently be replaced by a magnetohydrodynamic description without dissipation but having an anisotropic pressure; the so-called Chew, Goldberger, and Low (CGL) equation or the double adiabatic magnetohydrodynamic description.[1,2] The CGL equation, however, has a limited range of applicability and does not provide a good approximation when plasma motion varies appreciably along the magnetic lines of force. A rigorous mathematical treatment of this problem can be achieved by working with the coupled Vlasov and Maxwell equations in the limit $m/e \rightarrow 0$ which then exactly correspond to the drift approximation. This method of solution, given by Kruskal and Oberman,[3] will be called the quasi-magnetohydrodynamic description.

It is useful to note here the relationship that exists between ideal magnetohydrodynamics (MHD without dissipation), double adiabatic magnetohydrodynamics and quasi-magnetohydrodynamics.

Since these theories do not involve dissipation, stable equilibrium configurations are determined by the minima of the potential energy of the total system. Thus for a stable equilibrium there exists a potential energy W with the property that the variation δW of W is positive for all possible perturbation around the equilibrium condition (i.e., $\delta W > 0$). Denote the δW of the ideal, the double adiabatic, and the quasi-magnetohydrodynamic descriptions by δW_{MHD}, δW_{CGL}, and $\delta W_{\mathrm{quasi}}$, respectively. Then it has been proved by Bernstein[3a] and others[2–5] that for a system which is initially in a state of stable equilibrium and is subjected to an isotropic pressure there exists the inequality

$$\delta W_{\mathrm{MHD}} < \delta W_{\mathrm{quasi}} < \delta W_{\mathrm{CGL}}.$$

When the collision time is much greater than the growth rate of the instability the pressure does not necessarily remain isotropic even if it is so initially and as a result the use of the ideal magnetohydrodynamic description is not justified. However by virtue of this inequality, the system is stable in both the quasi-magnetohydrodynamic and the double adiabatic magnetohydrodynamic descriptions whenever it is stable in the ideal magnetohydrodynamic description. It should also be noted that stability

[1] G. F. Chew, M. L. Goldberger, and F. E. Low, The Boltzman equation and the one-fluid hydromagnetic equations in the absence of particle collisions. *Proc. Roy. Soc. (London)* **A236,** 112–118 (1956).

[2] M. N. Rosenbluth and N. Rostoker, Theoretical structure of plasma equations. *Phys. Fluids* **2,** 23–30 (1959).

[3] M. D. Kruskal and C. R. Oberman, On the stability of a plasma in static equilibrium. *Phys. Fluids* **1,** 275–280 (1958).

[3a] I. Bernstein, E. A. Frieman, M. D. Kruskal, and R. Kulsrud, *Proc. Roy. Soc. (London)* **A244,** 17–40 (1958). This paper is included in the present volume (p. 39).

[4] B. A. Trubnikov, Dynamical principle of stability for magnetohydrostatic systems. *Phys. Fluids* **5,** 184–191 (1962).

[5] R. Kulsrud, On the necessity of the energy principle of Kruskal and Oberman for stability. *Phys. Fluids,* **5,** 192–195 (1962).

in the double adiabatic magnetohydrodynamic description does not guarantee the stability of the quasi-magnetohydrodynamic description. This provides a justification for using the ideal magnetohydrodynamic description in the stability analysis of such systems.

Let us now assume that the motion of a plasma is governed by the hydromagnetic equations. Then, for an inviscid plasma, the equation of motion takes the form

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{1}{c}\,\mathbf{j} \times \mathbf{B} - \rho\,\nabla\phi, \tag{1}$$

in which ρ is the mass density, $\mathbf{v}$ is the flow velocity, p is the pressure, $\mathbf{j}$ is the current vector, $\mathbf{B}$ is the magnetic induction vector, ϕ is an external potential and where d/dt denotes the substantial or Lagrangian derivative $\partial/\partial t + \mathbf{v}\cdot\nabla$. Neglecting displacement current in the Maxwell equations gives the result[5a]

$$\frac{c}{4\pi}\,\nabla \times \mathbf{B} = \mathbf{j},$$

and consequently the right-hand side of Eq. (1) reduces to

$$-\nabla\left(p + \frac{\mathbf{B}^2}{8\pi}\right) + \frac{1}{4\pi}\,(\mathbf{B}\cdot\nabla)\mathbf{B} - \rho\,\nabla\phi. \tag{1'}$$

The second term of this expression can be interpreted as the magnetic tension acting as a restoring force against the bending of the magnetic lines of force. The electric field $\mathbf{E}$ is given by Ohm's law

$$\eta\mathbf{j} = \mathbf{E} + \frac{1}{c}\,[\mathbf{v} \times \mathbf{B}],$$

where η is the resistivity, by means of which the magnetic induction law, described by the second Maxwell equation, can be written as

$$\frac{\partial\mathbf{B}}{\partial t} - \nabla \times [\mathbf{v} \times \mathbf{B}] - \frac{c^2}{4\pi}\,\nabla \times [\eta\nabla \times \mathbf{B}] = 0.$$

If, moreover, $\eta = 0$ and the motion is also adiabatic, we have, besides the mass conservation law, the conservation laws for the entropy and for the magnetic flux,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{2a}$$

$$\frac{d}{dt}\,(p\rho^{-\gamma}) = 0, \tag{2b}$$

[5a] Gaussian units are used throughout, and we assume that the magnetic susceptibility is equal to unity.

and

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times [\mathbf{v} \times \mathbf{B}] = 0, \tag{2c}$$

while the electric field is given by the equation

$$\mathbf{E} + \frac{1}{c}[\mathbf{v} \times \mathbf{B}] = 0. \tag{3}$$

Equations (1), (2), and (3) constitute the fundamental equations for ideal magnetohydrodynamics.[6]

Suppose now that the plasma is in equilibrium. Then, from Eq. (1), we have the equilibrium condition,

$$\nabla p = \frac{1}{c}\,\mathbf{j} \times \mathbf{B}. \tag{4}$$

The boundary conditions may be classified as belonging to one of two types, which we shall denote by (I) and (II). Case (I) occurs when the plasma is surrounded by a vacuum magnetic field across the plasma-vacuum interface of which the mechanical pressure is balanced by the magnetic pressure. In case (II) the plasma is bounded by a conducting wall on which the pressure becomes zero. In case (I) the vacuum magnetic field is itself bounded by a conducting wall and, as a special case, the magnetic field may not exist inside the plasma. A mathematical expression of these boundary conditions has been given, for example, by Bernstein *et al.*[3a] Equation (4) implies that $\mathbf{B}$ and $\mathbf{j}$ are normal to ∇p and hence that they lie on an equipressure surface. Since the plasma boundary must be an equipressure surface and the magnetic field is solenoidal we find that the magnetic lines of force cover the equipressure surface. It has been proved that the closed equipressure surfaces on which the magnetic field nowhere vanishes must be topologically equivalent to a torus. At the same time it cannot be a simple torus but must be a twisted one, otherwise the magnetic lines of force must be helical.[7–9] An equilibrium configuration of this type is realized in the Stellarator, which has been extensively investigated by Spitzer and others.[10,11]

[6] For the details of the fundamental equations of magnetohydrodynamics see, for example, A. Jeffrey and T. Taniuti, "Nonlinear Wave Propagation with Applications to Physics and Magnetohydrodynamics," Chap. 4. Academic Press, New York, 1964.

[7] M. D. Kruskal and R. M. Kulsrud, Equilibrium of a magnetically confined plasma in a toroid. *Phys. Fluids* **1,** 265–274 (1958); and Ref. 6, Section 8.4.

[8] S. Hamada, Hydromagnetic equilibria and their proper coordinates. *Nucl. Fusion* **2,** 23–27 (1962).

[9] B. B. Kadomtsev, On equilibrium of a plasma under helical symmetry. *Soviet Phys. JETP* (*English Transl.*) **10,** 962–963 (1960).

[10] L. Spitzer, The stellerator concept. *Phys. Fluids* **1,** 253–264 (1958).

[11] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Some stable hydromagnetic equilibria. *Phys. Fluids* **1,** 281–296 (1958).

A discussion of the stability of the equilibrium characterized by Eq. (4) may be based on the linearized form of Eqs. (1) to (3). In terms of the small displacement vector $\boldsymbol{\xi}$, Eq. (1), when expressed in Lagrangian form, becomes

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}\{\boldsymbol{\xi}\}, \tag{5}$$

in which ρ_0 is the unperturbed density and $\mathbf{F}\{\boldsymbol{\xi}\}$ is a linear function with respect to $\boldsymbol{\xi}$ and its spatial derivatives. This result is in fact obtained by using Eq. (2) to express ρ, p, and $\mathbf{B}$ after an infinitesimal time Δt in terms of their equilibrium values and $\boldsymbol{\xi}$ at the start, and then introducing the resulting expressions into Eq. (1) and retaining only the first-order terms in $\boldsymbol{\xi}$. The explicit form of $\mathbf{F}\{\boldsymbol{\xi}\}$ has been given by Bernstein.[3a] If, for some displacement $\boldsymbol{\xi}$, $\mathbf{F}\{\boldsymbol{\xi}\}$ has the same direction as $\boldsymbol{\xi}$, then such a displacement is accelerated, and the perturbation grows, and so the equilibrium is unstable. This simple method for the determination of unstable modes is often useful and will sometimes be utilized in the subsequent explanations of instabilities.

Owing to the self-adjointness of $\mathbf{F}\{\boldsymbol{\xi}\}$ which follows directly from its explicit form,[3a] overstability, comprising an initial unstable oscillatory behavior, does not occur. That is, when $\boldsymbol{\xi}$ is given in the form $\boldsymbol{\xi} \sim e^{i\omega_n t}\boldsymbol{\xi}_n$, Eq. (5) reduces to a Sturm-Liouville type eigenvalue problem $-\omega_n^2\rho_0\boldsymbol{\xi}_n = \mathbf{F}\{\boldsymbol{\xi}_n\}$, the eigenvalues, ω_n^2, of which are real with negative values of ω_n^2, should they occur, corresponding to instability.

An alternative method of solution is to utilize the variational principle. When the plasma is in equilibrium the potential energy, say W, is given by the sum of the energy of the magnetic field, the internal energy, and the potential energy due to external forces. Namely, we have

$$W = \int \left\{ \frac{1}{8\pi} \mathbf{B}^2 + \rho U + \rho\phi \right\} d\tau, \tag{6}$$

where ϕ is the external potential, U is the internal energy density per unit mass, and where the integration is extended over the whole region occupied by the plasma. When the plasma may be assumed to be a polytropic gas, so that the internal energy is proportional to the temperature, the equation of state leads directly to the relationship

$$\rho U = \frac{p}{\gamma - 1}.$$

Let us now consider a perturbed state about an equilibrium condition in which the small displacement vector $\boldsymbol{\xi}$ is arbitrarily prescribed. The variations of ρ, p, and $\mathbf{B}$ denoted by $\delta\rho$, δp, and $\delta\mathbf{B}$, respectively, are not independent since they must satisfy the conservation laws (2), by means of which $\delta\rho$, δp, and $\delta\mathbf{B}$ may be expressed in terms of the equilibrium conditions ρ_0, p_0, and $\mathbf{B}_0$. Inserting these expressions into Eq. (6) results, to the first order in $\boldsymbol{\xi}$, in the equilibrium equation (4). Hence the nonvanishing next lowest order term is the second-order one which is quadratic in $\boldsymbol{\xi}$, which we now denote by δW. By analogy with the stability theory of particle mechanics, we assert that the system is stable if and only if $\delta W > 0$ for any $\boldsymbol{\xi}$. If δW becomes negative for some $\boldsymbol{\xi}$, the system is unstable. This can be proved rigorously by noting the identity[3a]

$$\delta W \equiv -\tfrac{1}{2}\int \boldsymbol{\xi}\cdot\mathbf{F}\{\boldsymbol{\xi}\}\, d\tau,$$

in which $\mathbf{F}\{\boldsymbol{\xi}\}$ is given by Eq. (5) and, as before, $d\tau$ denotes a volume element.

Under the boundary condition (I) the plasma-vacuum interface is also displaced. Hence by means of the Gaussian divergence theorem the divergence terms in the integrand of δW may be transformed into a surface integral. After some manipulation involving the boundary conditions we find that there are some terms in this surface integral that are dependent only on the vacuum field quantities. The surface integral of such terms can then be transformed further into a volume integral over the vacuum region. As a result the explicit form of δW comprises three parts, namely, δW_F, δW_S, and δW_V. Here δW_F is the volume integral over the unperturbed plasma region, δW_S is the surface integral over the unperturbed plasma boundary which is equal to the work done against the displacement of the boundary, and δW_V is the volume integral over the vacuum region which is equal to the change of the vacuum magnetic field energy resulting from the deformation of the plasma-vacuum interface. The explicit form of δW given by Bernstein[3a] is as follows:

$$\delta W = \delta W_F + \delta W_S + \delta W_V, \tag{7a}$$

with

$$\delta W_F \equiv \tfrac{1}{2}\int d\tau \left\{\frac{\mathbf{Q}^2}{4\pi} - \boldsymbol{\xi}\cdot(\mathbf{j}_0 \times \mathbf{Q}) + (\gamma p_0 \nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p_0)\nabla\cdot\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla\phi)\nabla\cdot(\rho_0\boldsymbol{\xi})\right\}, \tag{7b}$$

$$\delta W_S = \tfrac{1}{2}\int d\sigma\, (\mathbf{n}\cdot\boldsymbol{\xi})^2 \mathbf{n}\cdot\left\langle \nabla\left(\frac{\mathbf{B}_0^2}{8\pi} + p_0\right)\right\rangle, \tag{7c}$$

and

$$\delta W_V = \frac{1}{8\pi}\int \mathbf{B}_V^2\, d\tau, \tag{7d}$$

where $\mathbf{Q}$ is used to denote the expression

$$\mathbf{Q} = \nabla \times [\boldsymbol{\xi} \times \mathbf{B}_0],$$

in which the subscript 0 refers to the equilibrium condition, $\mathbf{n}$ is the unit normal to the interface, $\langle X \rangle$ denotes the jump in X across the interface in the direction $\mathbf{n}$, and $d\sigma$ denotes a surface element.

It is difficult to investigate the sign of δW for all $\boldsymbol{\xi}$, but it is often possible to find a class of displacements making δW negative (i.e., some type of instability). Generally speaking, a perturbation which changes the magnetic field has a tendency to increase the magnetic energy because of the effect of the magnetic tension in the lines of force which provides a restoring force. Hence if the magnetic energy is much greater than the internal energy a mode which changes the magnetic field has a tendency to make δW positive. Consequently it is those perturbations which do not change the magnetic field which are most likely to cause instability. In other words, in order to find an unstable mode, we should seek to find a displacement which does not disturb the magnetic field. In the following discussions we shall use heuristic arguments to discover unstable modes in an attempt to understand the possible mechanisms of instability experienced by a confined plasma.

We consider a plasma supported against gravity by a magnetic field. In equilibrium the plasma lies above the plane $x = 0$, say, in the half-space $x > 0$, and a uniform magnetic field is applied in the horizontal z direction in the vacuum occupying the lower half-space, $x < 0$. We shall assume that the magnetic field does not exist inside the plasma.

For simplicity we also assume that the plasma is incompressible and consequently that the density is everywhere constant. Then the equilibrium equation in the plasma takes the form

$$\frac{dp}{dx} = -\rho g,$$

and at the interface between the plasma and the vacuum we have the equation of pressure balance

$$p = \frac{B^2}{8\pi},$$

in which B is the strength of the vacuum magnetic field. Suppose the plasma is disturbed slightly in a direction normal to the magnetic field, as is shown in Fig. 1, in such a manner that the plasma located in the region P' interchanges position with the magnetic field occupying the region P of the same cross-sectional area. Since the magnetic lines of force are not bent by such a displacement the magnetic field energy is unchanged. On the other hand the pressure is increased in P by an amount $\rho g\,|\delta x|$. Hence the plasma is pushed down at P with the result that the perturbation grows. The system is therefore unstable. Since the wavy perturbation tends to grow to a flute shape an instability of this type is often referred to as a *flute-type instability*.

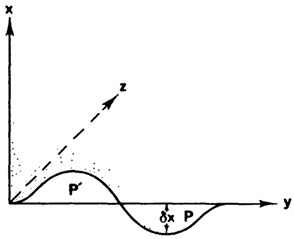

FIG. 1. Disturbed plasma-vacuum interface.

This result may also be deduced from the energy principle. Since the change of the internal energy of an incompressible plasma may be attributed to the work done by external forces, in the present case it may be expressed as the work done on the interface against the pressure gradient [cf. Eq. (7c)]:

$$\int d\sigma\ (\mathbf{n}\cdot\boldsymbol{\xi})^2\mathbf{n}\cdot\nabla p = -\int d\sigma\ (\mathbf{n}\cdot\boldsymbol{\xi})^2\rho\mathbf{n}\cdot\nabla\phi,$$

in which $\mathbf{n}$ is the normal to the interface pointing toward the plasma and ϕ is the gravitational potential. Since ϕ increases as x increases, the above integral is negative; therefore the system is unstable. (Since ϕ is linear in x the change of the potential energy inside the plasma is first order with respect to the magnitude of the displacement, and consequently it does not contribute to δW, which is second order.) The magnetic field does not play an essential role in this discussion, and it has been shown that the characteristic features of the instability are the same as those of the Rayleigh-Taylor instability in ordinary hydrodynamics in which, in the absence of a magnetic field, a fluid is supported against gravity by a fluid of smaller density.[12]

Let us obtain a rough estimate of the growth rate of the instability in this well-known example. Since the density is constant the perturbed equation of motion takes the form

$$\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\nabla\,\delta p,$$

[12] S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," §91. Oxford Univ. Press (Clarendon), London and New York, 1961.

while at the interface $\nabla\, \delta p = (\rho_1 - \rho_2)g = -\langle\rho\rangle g$ in which the subscripts 2 and 1 refer, respectively, to the upper and lower half-planes and $\langle\rho\rangle \equiv \rho_2 - \rho_1$. For an unstable mode $|\omega|$ may be considered to be the growth rate, namely, the e-folding time of the growing instability. On the fluid-fluid interface the equation of motion may be approximated by $-\omega^2\bar{\rho}\xi \approx \langle\rho\rangle g$ in which $\bar{\rho}$ is the mean density. Hence, putting $\xi \approx 1/k$, where k is the wave number of the perturbation, we have

$$|\omega^2| \approx gk\,\frac{\langle\rho\rangle}{\bar{\rho}}. \tag{8a}$$

In the limit as the density of the lighter fluid tends to zero our problem tends to the case of a plasma supported by a vacuum magnetic field, and we have

$$|\omega^2| \approx gk, \tag{8b}$$

which is identical to the result of the exact calculation performed by Kruskal and Schwarzschild.[12a]

So far we have assumed that the magnetic field does not exist inside the plasma, but even if this is not the case we still have the same result whenever the magnetic field is uniform and parallel to the field outside the plasma. This is due to the fact that for perturbations normal to the magnetic field, of the type illustrated in Fig. 1, the magnetic lines of force inside as well as outside the plasma are not bent, and consequently the magnetic energy does not change. This may be seen analytically as follows. The magnetic field induced by the motion of the plasma across the applied magnetic field is given by $\nabla \times [\mathbf{v} \times \mathbf{B}]$, where $\mathbf{B}$ is the applied magnetic field and $\mathbf{v}$ is the velocity of the plasma [cf. Eq. (2c)]. However, for the present configuration in which the perturbation is independent of z, while $\mathbf{B}$ is directed along the z axis, we have $\partial\mathbf{B}/\partial t = -\mathbf{B}(\nabla\cdot\mathbf{v}) - (\mathbf{v}\cdot\nabla)\mathbf{B}$, which for an incompressible fluid reduces to $(\partial/\partial t + \mathbf{v}\cdot\nabla)\mathbf{B} = 0$, implying that the magnetic field is carried unchanged with each fluid element. Hence the magnetic energy obtained by integrating over the totality of fluid elements is unchanged. Moreover if $\mathbf{B}$ is uniform over the whole space, as is the case in which a heavier fluid is supported by a lighter one, then the magnetic field does not change everywhere. The growth rate of the instability is again given by Eq. (8a).[12b] We should note however that even if the plasma is not displaced in the z direction (i.e., v_z is zero), the expression will not be zero if $\mathbf{v}$ is a function of z; namely, if the displacement transverse to the magnetic field varies along the magnetic field.

[12a] M. D. Kruskal and M. Schwarzschild, *Proc. Roy. Soc. (London)* **A223,** 348–360 (1954). This paper is included in the present volume (p. 63).

[12b] S. Chandrasekhar, ftn. 12, §97, p. 464.

In these examples the magnetic fields act only to maintain the pressure balance in equilibrium, and they do not provide any stabilizing effect. However, as shown in Eq. (1′), besides an isotropic pressure the magnetic field provides a tension which acts as a restoring force against the bending of lines of force and so exerts some stabilizing effect. In fact, if a perturbation has a component along the magnetic field as well as the transverse component considered so far, the system is stable for wavelengths shorter than some critical one dependent on the magnetic field strength.[12c] In this case the interface in Fig. 1 is disturbed not only along the y-direction but also along the z direction, so that the lines of force are necessarily bent and the magnetic tension acting along the lines of force produces a restoring force which opposes this bending. This restoring force is obviously more effective for distortions of larger curvature as, for example, in the case of shorter wavelength perturbations along the magnetic field. The modification to Eq. (8) resulting from this effect may be estimated roughly as follows.

Let the plasma be displaced as before in a direction transverse to the magnetic field, but let the displacement be modulated along the magnetic

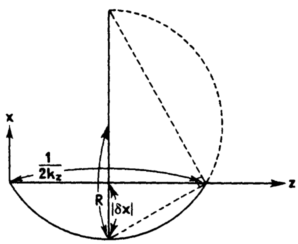

FIG. 2.

field (i.e., ξ_y is a function of z). The magnetic tension resulting from $(\mathbf{B}\cdot\nabla)\mathbf{B}$ in Eq. (1′) may be expressed as B^2/R, where the radius of curvature R of the line of force is given by the equation

$$2R\,|\delta x| \approx \left(\frac{1}{2k_z}\right)^2 \qquad \text{or, approximately,} \qquad R \approx \frac{1}{k_z^{\,2}}\,\frac{1}{|\delta x|},$$

in which k_z is the z component of the wave number vector of the perturbation, B is the strength of the magnetic induction vector inside the plasma, and where we have assumed $|\delta x| \ll R$ (cf. Fig. 2). Hence we have

$$-\omega^2\bar{\rho}\,|\delta x| \approx \langle\rho\rangle g - B^2 k_z^{\,2}\,|\delta x|.$$

[12c] S. Chandrasekhar, ftn. 12, §97.

Putting $\delta x \approx 1/k$ yields

$$-\omega^2 \approx gk \left\{ \frac{\langle \rho \rangle}{\bar{\rho}} - \frac{B^2 k_z^2}{\bar{\rho} g k} \right\}, \tag{8c}$$

which shows that for sufficiently large k_z the system is stable. This expression is correct, apart from numerical factors, provided the magnetic fields inside and outside the plasma are the same.[12c] For a plasma bounded by a vacuum the equation obtained by letting the smaller density tend to zero is still valid within an order of magnitude.

It is interesting to note that the effect of the magnetic tension is the same as that of the surface tension in the Rayleigh-Taylor instability without a magnetic field. Namely, if at the fluid-fluid interface the surface tension T_{eff} is introduced, the growth rate for such a system may be derived from Eq. (8c) by replacing $(B/R^2)\mathbf{k}\cdot\mathbf{B}$ by T_{eff} in which $\mathbf{k}$ is the wave number vector of the perturbation.[12a] Alternatively, introducing the Alfvén frequency ω_A through the equation

$$\omega_A^2 = \frac{B^2}{4\pi\bar{\rho}} k_z^2,$$

we may write Eq. (8c) in the form

$$\omega^2 \approx -gk + \omega_A^2, \tag{8c$'$}$$

which shows that for short wavelengths the disturbance propagates as an Alfvén wave while for long wavelengths the instability mode is dominant. From these results we may conclude that the transverse perturbation characterized by $\mathbf{k}\cdot\mathbf{B} = 0$ is the one most likely to produce instability in the system under consideration. The results also suggest a possibility of stabilizing the Rayleigh-Taylor instability. For example, let the lines of force of the uniform magnetic field be fixed at both ends to perfectly conducting plates. Then, even if the plasma is displaced normal to the magnetic field, the magnetic lines of force are necessarily bent, and we have a situation similar to the one just discussed (cf. Fig. 3). Hence Eq.

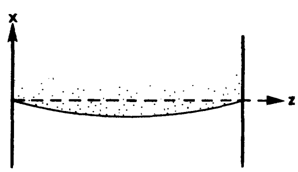

FIG. 3. Rayleigh-Taylor instability.

(8c) is still valid to within an order of magnitude when k_z^{-1} is replaced by the distance between the plates and $\langle \rho \rangle = 2\bar{\rho}$ is set equal to ρ. Taking into account the change of the vacuum magnetic field caused by the bending of the boundary lines of force and the resulting change in the magnetic pressure, Vedenov and others[13] derived the exact stability condition.

An alternative method of stabilization of the Rayleigh-Taylor type instability could be achieved by shearing the lines of force. Let the horizontal uniform magnetic fields inside and outside the plasma no longer be parallel so that the magnetic field has a shear across the interface. Then the interchange illustrated in Fig. 1 is prohibited, and any deformation of the interface produces a bending of the lines of force. For example, suppose that the magnetic field inside the plasma has only the y component B_0, say, while the vacuum field outside the plasma has the z component B_{z0} as well as the same y component B_0, and consider a perturbation in the (x, y) plane, independent of z. If B_0 is zero, the configuration is the same as that shown in Fig. 1, and the equilibrium is unstable. However, in the presence of the y component of the magnetic field, such a perturbation necessarily bends the lines of force along the y axis, and we again have the same magnetic tension effect that was discussed previously in Eq. (8c), though k_z is now replaced by k_y.

In the foregoing discussions we have neglected all the dissipative terms such as finite electrical resistivity and viscosity, since they are usually small outside the region in which the physical quantities vary rapidly and even in such a region they are effective only for short wavelengths. In some cases these effects can give rise to a new type of instability as has been recently discussed by Furth *et al.*,[13a] who showed that resistivity acts in a manner that destroys the shear stabilization just mentioned.[14] In order to understand the role played by resistivity in this destabilization process let us first consider Ohm's law:

$$\eta \mathbf{j} = \mathbf{E} + \frac{1}{c}[\mathbf{v} \times \mathbf{B}]. \tag{9}$$

The right-hand side is equal to the electric field in the coordinate system moving with the plasma and is equal to zero if η is zero; namely, the first term, $\mathbf{E}$, which is the electric field in the laboratory system, changes so as to cancel out the induced field given by the second term. This electric field $\mathbf{E}$ results from the induction of the magnetic field in such a manner that the lines of force are frozen into the plasma. If η is

[13] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Stability of a plasma. *Soviet Phys.—Usp.* (*English Transl.*) **4,** 332–369 (1961).

[13a] H. P. Furth, J. Kileen, and M. N. Rosenbluth, *Phys. Fluids* **6,** 459–484 (1963). This paper is included in the present volume (p. 77).

[14] J. D. Jukes, Stability of the sharp pinch and unpinch with finite conductivity. *Phys. Fluids* **4,** 1527–1533 (1961). See also K. V. Roberts and J. B. Taylor, *Phys. Fluids* **8,** 315–323 (1965).

small but finite, $\mathbf{E}$ no longer exactly cancels out the second term and a slight defreezing of the lines of force occurs. The degree to which $\mathbf{E}$ then compensates the second terms depends, naturally, on the size of η. In the other extreme when η is very large $\mathbf{E}$ will not change and the motions of the plasma and of the lines of force become detached. Hence Eq. (9) reduces to

$$\eta \mathbf{j}_1 \approx \frac{1}{c}[\mathbf{v} \times \mathbf{B}_0], \tag{9'}$$

where $\mathbf{B}_0$ is the equilibrium magnetic field and $\mathbf{j}_1$ is the first-order perturbation of the current $\mathbf{j}$. This then produces a Lorentz force $\mathbf{F}_s = \mathbf{j}_1 \times \mathbf{B}_0$ which acts on the plasma. Since

$$\mathbf{F}_s = \mathbf{j}_1 \times \mathbf{B}_0 = \frac{1}{\eta}[\mathbf{B}_0(\mathbf{v}\cdot\mathbf{B}_0) - \mathbf{v}B_0^2],$$

the force opposes the fluid motion, and it becomes stronger as $\eta \to 0$. This means that for small η a strong force is necessary if the plasma is to be detached from lines of force so that $\mathbf{E}$ and consequently $\mathbf{B}$ do not change. However if $\mathbf{B}_0$ has a null point, this restoring force becomes arbitrarily weak in the neighborhood of that point and may be dominated there by other forces acting to accelerate the fluid motion. Analysis shows that, even if η is not large, Eq. (9′) is still approximately true in regions where $\mathbf{B}_0$ is small.

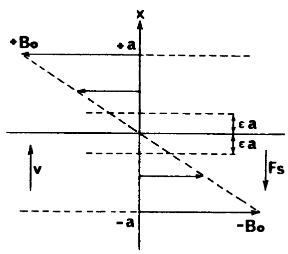

FIG. 4.

As an illustrative example, consider a current layer of width $2a$, the magnetic field in which is given by (see Fig. 4)

$$B_{x0} = B_{z0} = 0 \tag{10a}$$

and

$$B_{y0} = x, \qquad -a \leq x \leq a, \tag{10b}$$

resulting in the current density

$$j_{x0} = j_{y0} = 0 \tag{11a}$$

and

$$j_{z0} = 1. \tag{11b}$$

Equations (10) imply that the magnetic field reverses its direction in crossing the layer, and so it must experience a 180° shear. We note that, for finite resistivity η, the equation for the equilibrium magnetic field takes the form

$$\nabla \times [\eta \nabla \times \mathbf{B}_0] = 0$$

which is satisfied by Eqs. (10) if η is constant.

It can be shown[13a] that Eqs. (10) may be used to represent the magnetic field without any loss of generality. Namely, even for a more general twisted magnetic field such as that illustrated in Fig. 5 which does not

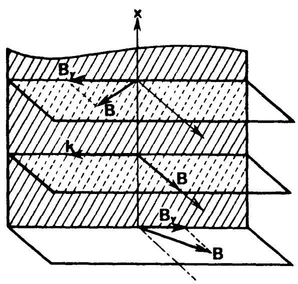

FIG. 5. Twisted magnetic field.

have a null point, a wave number vector **k** can always be chosen for the perturbation such that ($\mathbf{k} \cdot \mathbf{B}$) becomes zero at any desired point. The success of this argument is due to the fact that the component of the magnetic field normal to vector **k** is ignorable, and so the system of equations governing the perturbed field and flow can be reduced to a problem in the (k, x) plane involving the vector **k** which is perpendicular to the magnetic field at $(\mathbf{k} \cdot \mathbf{B}) = 0$. Hence, if the k axis is identified with the y axis of Fig. 4, we obtain the configuration that was previously examined.

The plasma will be assumed to be incompressible but to have an inhomogeneous density distribution, the change of which is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho = 0. \tag{2a'}$$

From Eqs. (10) the null point of the magnetic field is given by $x = 0$ in the neighborhood of which the detached motion is realized. This small region may be specified as $-\epsilon a \leq x \leq \epsilon a$.

Let the plasma be initially in equilibrium under the action of gravity, with the magnetic field so prescribed, and then suppose that the plasma moves upward in the positive x direction, as shown in Fig. 4. The vector $\mathbf{F}_s$ is then directed downward and acts as restoring force. However if the density of the plasma increases upward, an element of the low density being carried upward decreases the local density at an upper level, say by an amount $|\delta\rho|$. This change of density gives rise to a perturbation of the gravitational force of an amount $g\delta\rho$. Since $\delta\rho$ is negative the perturbation force thus produced is directed locally upward so that it acts to accelerate the motion. For a sufficiently small ϵa this force becomes greater than the restoring force $\mathbf{F}_s$ since the latter is only of order $B_{y0}^2 \approx (\epsilon a)^2$ thereby leading to the instability. Since this instability is realized by an interchange between heavier and lighter elements, this form of instability is sometimes referred to as the *interchange mode*. However the origin of the instability differs from that of the interchange instability previously considered.

Instead of the variation of density, a small spatial variation of resistivity may be assumed, in which case an element of low resistivity is carried upward, decreasing the local resistivity at an upper level. This change of resistivity gives rise to a perturbation current resulting in a Lorentz force

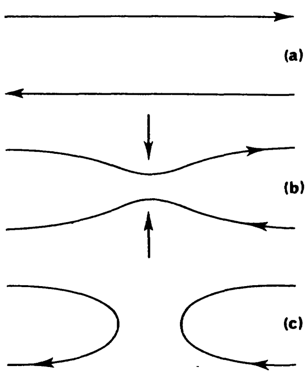

FIG. 6. The tearing mode.

which acts to accelerate the displacement. This mode of instability is often called the *rippling mode*. In connection with these modes we should also mention the *tearing mode* in which two magnetic lines of force, as shown in Fig. 6(a), are deformed as illustrated in Fig. 6(b) and are then torn into two loops such as those in Fig. 6(c). This type of instability was originally found by Dungey[15] for an X-type neutral point of a magnetic field. For a

[15] J. W. Dungey, "Cosmic Electrodynamics," pp. 98–102. Cambridge University Press, London and New York, 1958.

detailed discussion of these modes of instability we refer to the paper by Furth *et al.*[13a]

Let us now consider another type of ripple mode which was found by Kadomtsev[16,17] in connection with an anomalous diffusion process in the positive column discharge. Let the electric and the magnetic fields be applied in the positive z direction and assume that the temperature increases in the direction of negative x so that the resistivity increases in the direction of positive x. Now suppose that a rectangular volume of plasma immersed in the plasma occupying the whole space moves slightly in the direction of positive x, as shown in Fig. 7. The plasma of low resistivity is then carried in the positive x direction, so that at one end of the plate on the side of the positive x axis the local resistivity is decreased. Hence, when the plate is inclined to the direction of the applied electric field (the z axis), the decrease in resistivity will cause a surplus current to flow through the plate. As a result charge is accumulated on both faces of the plate giving rise to the electric field $\mathbf{E}'$ directed in the positive y direction. Therefore the drift $\mathbf{E}' \times \mathbf{B}/B^2$ will lead to instability. In the anomalous diffusion process, which was found by Hoh and Lehnert[18] to take place in the positive column discharge, the configuration is more complicated and needs elaborate study. However the essential mechanism, which is caused by a screw instability of the positive column, can still be understood by analogy with the ripple instability discussed here. A detailed analysis is

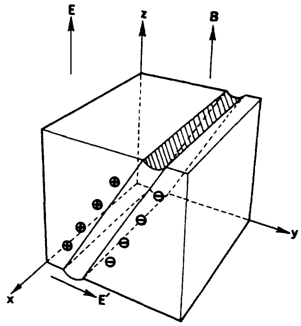

FIG. 7.

[16] B. B. Kadomtsev, Convection of the plasma of a positive column in a magnetic field. *Soviet Phys.—Tech. Phys.* (*English Transl.*) **6,** 927–933 (1962).

[17] B. B. Kadomtsev, Criteria for instability and gain. *Soviet Phys.—Tech. Phys.* (*English Transl.*) **6,** 882–888 (1962).

[18] F. C. Hoh and B. Lehnert, Diffusion processes in a plasma column in a longitudinal magnetic field. *Phys. Fluids* **3,** 600–607 (1960).

to be found in the paper by Kadomtsev and Nedospasov[18a] while for an intuitive explanation we refer to the papers by Hoh and Lehnert.[18–20]

Instead of using a magnetohydrodynamic description it is also possible to understand the instability mechanisms so far discussed on the basis of particle behavior. Explanations of this type have been given by Rosenbluth and Longmire.[20a] For example, consider the first system to be studied in which the plasma occupied the half-space $x > 0$ and was supported below against gravity by a vacuum magnetic field. When the magnetic field does not exist inside the plasma the only force acting on a particle is that due to gravity. Hence, under the perturbation that was illustrated in Fig. 1, particles which meet the interface at a point on the convex part P will, on the average, have somewhat greater velocity than those impinging at P', so that they exert a higher pressure on P. In this case we require that a particle which is reflected from a point of P should not arrive directly at another point of P without first being reflected from some other part of the interface. Namely, the wavelength of the perturbation must be less than the distance traversed by particles along the interface without undergoing a reflection.

When a uniform magnetic field exists inside a plasma, gravitational forces cause the electrons and ions to drift in opposite directions. Let the magnetic field be $\mathbf{e}_z B_0$ and the gravitational acceleration be $-\mathbf{e}_x g$, in which $\mathbf{e}_x$ and $\mathbf{e}_z$ are the unit vectors along the x and the z axes, respectively. Then the drift velocity for a particle of mass m and charge e is given by

$$-\frac{mc}{e}\frac{g\mathbf{e}_x \times B_0\mathbf{e}_z}{B_0^2}.$$

For the configuration shown in Fig. 1 this drift will result in charge separation in a surface layer of the plasma, as indicated in Fig. 8. Since the drift velocity is proportional to mass, the electron drift is much smaller than the ion drift, and may be neglected. The charge separation thus produced causes an electric field which then induces a further plasma drift in the negative x direction with the velocity $[\mathbf{E} \times B_0\mathbf{e}_z]/B_0^2$. The perturbation therefore grows. This discussion illustrates that the instability is to be attributed to charge separation on the plasma surface. Hence if the plasma

[18a] B. B. Kadomtsev and A. V. Nedospasov, *J. Nucl. Energy: Pt. C* **1**, 230–235 (1960). This paper is included in the present volume (p. 103).

[19] F. C. Hoh and B. Lehnert, Screw instability of a plasma column. *Phys. Rev. Letters* **7**, 75–76 (1961).

[20] F. C. Hoh, Screw instability of a plasma column. *Phys. Fluids* **5**, 22–28 (1962).

[20a] M. N. Rosenbluth and C. L. Longmire, *Ann. Phys.* (*N.Y.*) **1**, 120–140 (1957). This paper is included in the present volume (p. 109).

is bounded by conducting plates at both ends, as was illustrated in Fig. 3, the space charge will move along the lines of force and be short-circuited at the plates with a resultant stabilizing effect.

This also indicates that any force $\mathbf{F}$ that is independent of charge will cause a similar type of instability if it acts towards the vacuum and normal to the field, since it results in the drift velocity $(mc/e)[\mathbf{F} \times B_0\mathbf{e}_z]/B_0^2$ which again produces charge separation. For instance, consider a plasma

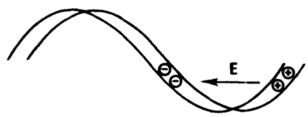

FIG. 8. Charge separation.

moving with an acceleration α and being pushed at one end by a magnetic piston (i.e., by a magnetic pressure resulting from an external vacuum magnetic field applied in a direction normal to the motion of the plasma). Then, in the reference system moving with the boundary, an inertia force $-\alpha$ acts on a unit mass of the plasma. If this force is directed toward the vacuum the inertia force has exactly the same effect as gravity, and the system is unstable. An instability of this type has been observed in the dynamic process of pinched discharges.[21,22]

By similar reasoning it can be shown that a plasma undergoing rotation is also unstable, in which case the effective force corresponding to gravitation is the centrifugal acceleration v^2/r, where v is the circumferential velocity of a particle situated at a radial distance r from the center of rotation. When such a rotation is realized by crossed electric and magnetic fields g must be replaced by the substitution

$$g \longrightarrow \frac{v^2}{r} = \frac{c^2E^2}{rB^2},$$

and consequently the growth rate $\sqrt{gk}$ takes the form

$$\sqrt{\left(\frac{k}{r}\right)}\frac{cE}{B}.$$

[21] D. J. Albares, N. A. Krall, and C. L. Oxley, Rayleigh-Taylor instability in a stabilized linear pinch tube. *Phys. Fluids* **4**, 1031–1039 (1961).

[22] T. S. Green and G. B. F. Niblett, Rayleigh-Taylor instabilities of a magnetically accelerated plasma. *Nucl. Fusion* **1**, 42–46 (1960).

A detailed theoretical discussion of this point has been given by Gerjuoy and Rosenbluth and also by Vedenov *et al.*[13,23] Experimental observations have been reported by Rostoker and Kolb.[24,25]

Curvature of the magnetic field in a plasma also gives rise to an instability if the center of curvature lies inside the plasma. In this case the force corresponding to gravitation results from the motion of charged particles along the curved magnetic lines of force. Let the velocity component of a particle along and perpendicular to a line of force be $v_{\parallel}$ and $v_{\perp}$, respectively, and let its magnetic moment be μ. Then the centrifugal force acting on a particle of mass m which is directed normal to the field is given by

$$\frac{mv_{\parallel}^2}{R^2}\,\mathbf{R},$$

where $\mathbf{R}$ is the radius-of-curvature vector pointing from the center of the curvature to a point on the line of force. As well as this inertia force there is also a magnetic force $-\mu\,\nabla B = -\frac{1}{2}(\mu/B)\,\nabla B^2$ acting on the particle. If the plasma pressure is so low that the magnetic field approximates the vacuum field, the following identity is true:

$$\tfrac{1}{2}\,\nabla\mathbf{B}^2 = \mathbf{B}\times[\nabla\times\mathbf{B}] + (\mathbf{B}\cdot\nabla)\mathbf{B} = -B^2\frac{\mathbf{R}}{R^2} + \frac{\mathbf{B}}{B}(\mathbf{B}\cdot\nabla)B. \tag{12}$$

Adding the two effects gives for the force on the particle normal to the field

$$(mv_{\parallel}^2 + \tfrac{1}{2}mv_{\perp}^2)\,\frac{\mathbf{R}}{R^2} = \frac{P_{\parallel} + P_{\perp}}{\rho}\,\frac{\mathbf{R}}{R^2}, \tag{13}$$

where $P_{\parallel}$ and $P_{\perp}$ are the components of the pressure tensor along and perpendicular to the line of force. This expression at once leads to an effective gravitational acceleration

$$g_{\text{eff}} \sim \frac{P_{\parallel} + P_{\perp}}{\rho R}. \tag{14}$$

Therefore if $\mathbf{R}$ is everywhere directed toward the vacuum so that the center of the curvature lies inside the plasma, thereby making the plasma boundary everywhere convex to the vacuum, the system is unstable.

[23] E. Guerjuoy and M. N. Rosenbluth, Pinch with a rotating plasma. *Phys. Fluids* **4,** 112–122 (1961).

[24] N. Rostoker and A. C. Kolb, Fission of a hot plasma. *Phys. Rev.* **124,** 965–969 (1961).

[25] N. Rostoker and A. C. Kolb, Rotation of plasmas in θ pinches. *Phys. Fluids* **5,** 741–742 (1962).

In all these cases the effective gravitational acceleration g_{eff} is usually small, and consequently the growth rate $\omega = \sqrt{gk}$ is much less than the ion cyclotron frequency. It has been pointed out by Rosenbluth and others [25a] that in such a case the instabilities can be stabilized by the effect of a finite ion Larmor radius which, in the drift approximation or in the equivalent magnetohydrodynamic approximation, has been assumed to be zero.

Let us return to the system illustrated in Fig. 8 and consider the drift caused by the electric field due to the charge separation on the plasma boundary. Suppose that the charge separation takes place as shown in Fig. 9 and that the spatial dependence is approximately of the form $\sin ky$. (Note that the magnitude of the charge separation is out of phase with the displacement of the surface which takes the approximate form $\cos ky$.)

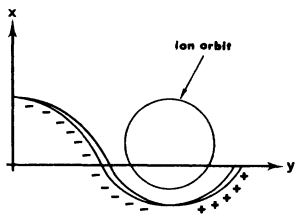

FIG. 9.

From the Poisson equation it follows immediately that $E_y \approx \cos ky$. Since the ions have a much greater Larmor radius than the electrons, as can be seen from Fig. 9, they experience a weaker electric field than do the electrons. On an average the difference is given by

$$(r_{iL}\nabla)^2 E_y = -r_{iL}^2 k^2 E_y,$$

where r_{iL} is the ion Larmor radius. As a result the drift velocity of the ions is less than that of the electrons, and so a charge separation current $\mathbf{j}'$ flows in the negative x direction. The order of magnitude of this current may be estimated as

$$|en(\mathbf{v}_{Di} - \mathbf{v}_{De})| \approx r_{iL}^2 k^2 necE_y/B,$$

where $\mathbf{v}_{Di}$ is the drift velocity of the ions in the weaker electric field, $\mathbf{v}_{De}$ is the drift velocity of the guiding center (i.e., of the electrons) and n is the equilibrium ion density.

On the other hand the original current $\mathbf{j}$ due to the gravitational drift is equal to $e\mathbf{v}_{DG}\,\delta n$, where $\mathbf{v}_{DG}$ is the ion drift velocity due to gravity and is

[25a] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, *Nucl. Fusion: Suppl.* No. 1, 143–150 (1962). This paper is included in the present volume (p. 131). See also A. B. Mikkailovskiy, *Nucl. Fusion* **4,** 108–114 (1964).

equal to $\mathbf{g} \times \mathbf{B}/\omega_{ic}B$. Here ω_{ic} is the ion cyclotron frequency and δn is the density perturbation resulting from the hydrodynamic motion of the plasma. Since the hydrodynamic motion is caused by the drift velocity $c[\mathbf{E} \times \mathbf{B}]/B^2$ (in the lowest order of perturbation the drifts accompanying the charge separation do not contribute to the mass motion of plasma but only to the current), δn is given by the mass conservation condition

$$|\omega|\,\delta n = -c\left(\frac{E_y}{B}\right)kn,$$

where $|\omega|$ is the growth rate of the instability and is equal to $\sqrt{gk}$. Consequently we have

$$|\mathbf{j}| \approx \frac{|\omega|}{\omega_{ic}}\,nec\left(\frac{E_y}{B}\right).$$

Therefore $|\mathbf{j}|$ becomes comparable in magnitude to $|\mathbf{j}'|$ if

$$\frac{|\omega|}{\omega_{ic}} \approx (kr_{iL})^2,$$

which is valid for a sufficiently slow growth rate. When this condition is realized, the acceleration is significantly altered so that the instability may be suppressed. An intuitive explanation has been attempted by Hoh.[26]

It has been shown by Roberts and Taylor[27] and by Rudakov[28] that the finite Larmor radius effect can also be achieved by introducing terms taking forms similar to those of viscosity into the hydromagnetic equation. Even in the collisionless approximation the required terms are obtained as higher order corrections to the CGL approximation. It should, however, be noted that these terms do not lead to any dissipative effects and that they are essentially different in nature from the true viscous effect. The finite Larmor radius effect on the gravitational instability in a plasma with finite resistivity has been discussed by Jukes.[29]

It has already been shown [cf. Eq. (13)] on the basis of orbit theory that a plasma which is confined by a magnetic field is unstable if its surface is convex to the vacuum. We now discuss this result from the hydromagnetic

[26] F. C. Hoh, Simple picture of the finite Larmor radius stabilization effect. *Phys. Fluids* **6,** 1359 (1963).

[27] K. V. Roberts and J. B. Taylor, Magnetohydrodynamic equations for finite Larmor radius. *Phys. Rev. Letters* **8,** 197–198 (1962).

[28] L. I. Rudakov, Influence of plasma viscosity in magnetic field on stability of plasma (in Russian). *Nucl. Fusion* **2,** 107–108 (1962).

[29] J. D. Jukes, Gravitational resistive instabilities in a plasma with a finite Larmor radius. *Phys. Fluids* **7,** 52–58 (1964). See also B. Coppi, *Phys. Fluids* **7,** 501–517 (1964).

view point. Let us first consider the case in which no magnetic field exists inside the plasma. Suppose that the plasma-vacuum interface is displaced locally toward the vacuum in such a way that the internal energy of the plasma does not change. If the vacuum magnetic field increases with increasing distance from the plasma boundary then, as the plasma moves into the vacuum, the magnetic pressure on the interface increases. If we assume that the mechanical pressure of the plasma remains constant then the resultant force produced by the magnetic pressure gradient is in the direction opposite to the displacement and so acts as a restoring force. Therefore if at all points of the interface the vacuum magnetic pressure increases with distance from the plasma boundary the system is stable. On the other hand if it decreases with increasing distance from the boundary, the magnetic pressure on the boundary becomes weaker as the interface is displaced toward the vacuum and the perturbation can grow. Of course in this case the change of the vacuum magnetic field due to the perturbation must be taken into account since it would increase the magnetic pressure when the vacuum field is bounded by a conducting wall. However, for displacements normal to the magnetic field, the increase in the magnetic pressure due to the perturbation is usually less than the radial attenuation of the unperturbed magnetic pressure.

The explanation may be understood more clearly if we note the similarity to a gravitational potential. In the present case the interface is not planar, and consequently the unperturbed magnetic field is not uniform and so exerts a pressure. The instability is caused by the change of this magnetic pressure on the interface resulting from the difference in the magnetic potential. In either case the lines of force exert a restoring force against bending, and consequently perturbations which interchange magnetic lines of force are most likely to result in instability.

The results may also be considered from the variational standpoint. Since the unperturbed P_0 is constant, choosing $\boldsymbol{\xi}$ to satisfy $\nabla\cdot\boldsymbol{\xi} = 0$ we immediately find that δW_F is equal to zero and that δW_S reduces to

$$\delta W_S = -\tfrac{1}{2}\int d\sigma \left\{(\mathbf{n}\cdot\boldsymbol{\xi})^2(\mathbf{n}\cdot\nabla)\frac{\mathbf{B}_0{}^2}{8\pi}\right\},$$

where $\mathbf{n}$ is the unit normal to the interface pointing toward the plasma and $\mathbf{B}_0$ is the unperturbed vacuum magnetic field. From this equation it follows at once that if $(\mathbf{n}\cdot\nabla)|\mathbf{B}_0|^2 < 0$, and so the magnetic pressure increases with distance from the interface, then δW_S is positive and consequently $\delta W > 0$ (even if $\nabla\cdot\boldsymbol{\xi} \neq 0$) and the system is stable. (Note that $\delta W_V > 0$.) Recalling Eq. (12) also implies that the system is stable if the plasma interface is everywhere concave toward the vacuum. On the other hand if the interface is convex δW_S becomes negative, since it can be shown that[3a] the positive contribution of δW_V can be made negligible when compared with $|\delta W_S|$, thereby showing that the system is unstable. As in the gravitational instability, the wavy deformation will grow to a flute along the lines of force. The stable configuration in which the plasma is everywhere concave toward the vacuum can be realized by a cusped magnetic field. Confinement by a cusp-type magnetic field has been extensively investigated by Grad *et al.*, who proved that the system is stable even against perturbations of finite amplitude.[30–32] The disadvantage of this system is that the cusp is leaky.

We now proceed to a discussion of the case in which a magnetic field exists in the plasma. Suppose that the field has no shear, so that interchange of lines of force is possible in the manner considered in the case of a gravitational instability. In this new problem, however, the lines of force are no longer uniform but are curved, and we shall now need to give a more elaborate explanation. With Rosenbluth and Longmire,[32a] let us assume that the plasma pressure is so small compared to the magnetic pressure that the magnetic field may be assumed to be nearly equal to the vacuum field. Any distortion of the field then increases its energy since the vacuum field is the lowest energy state.[32a] Hence we again see that a perturbation which leaves the magnetic energy unchanged is one which is most likely to lead to instability. In order to find the displacement let us consider the change of the magnetic field energy caused by the interchange of two flux tubes which we shall denote by I and II. Without loss of generality we shall consider a flux tube of a constant cross section, say A. Then the magnetic energy in the tube is equal to

$$\frac{1}{8\pi}\int B^2\,d\tau \sim \frac{1}{8\pi}B^2AL = \frac{1}{8\pi}\frac{\phi^2L}{A},$$

in which L is the length of the tube and ϕ is the flux which is defined by the relation

$$\phi = BA.$$

Suppose now that the plasma occupying flux tube I moves into tube II while the plasma occupying flux tube II moves into tube I, so that an

[30] J. Berkowitz, H. Grad, and H. Rubin, Magnetohydrodynamic stability. *Proc. U.N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958* Vol. 31, pp. 177–189. Columbia Univ. Press (I. D. S.), New York, 1959.

[31] J. Berkowitz, K. O. Friedrichs, H. Goertzel, H. Grad, J. Killeen, and H. Rubin, Cusped geometrics. *Proc. U.N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958* Vol. 31, pp. 171–176. Columbia Univ. Press (I. D. S.), New York, 1959.

[32] H. Grad, Containment in a cusped plasma system. *Progr. Nucl. Energy, Ser. XI* **2,** 189–200 (1963). Also see the references in this paper.

[32a] This can be seen from the fact that the variation of the magnetic energy $\delta\int \mathbf{B}^2\,dv = 0$ together with the condition $\nabla \cdot \mathbf{B} = 0$ leads to the equation $\nabla \times \mathbf{B} = 0$.

interchange between the two flux tubes occurs. Since the magnetic flux is conserved as the plasma moves, in the final state, the magnetic flux in tube I is equal to that of the plasma which was initially in II. We denote this by ϕ_{II} and similarly the flux in tube II becomes ϕ_I. Hence the change in the magnetic energy is given by

$$\frac{1}{8\pi}\left\{\frac{\phi_I^2 L_{II}}{A_{II}} + \frac{\phi_{II}^2 L_I}{A_I}\right\} - \frac{1}{8\pi}\left\{\frac{\phi_{II}^2 L_{II}}{A_{II}} + \frac{\phi_I^2 L_I}{A_I}\right\}.$$

Therefore if $\phi_I = \phi_{II}$, namely, if the fluxes are equal, the magnetic energy does not change. In this case the difference between the volumes of the two tubes results in a change in the final pressure. Denoting the final pressure in the tubes I and II by p'_I and p'_{II}, respectively, we then have the following expression for the change in the internal energy,

$$\delta W = \frac{1}{\gamma - 1}\{(p'_I V_I + p'_{II} V_{II}) - (p_I V_I + p_{II} V_{II})\}. \tag{15}$$

On the other hand, the adiabatic compression law,

$$pV^\gamma = \text{constant}$$

yields the relations

$$p'_I = (p_{II} V_{II}^\gamma)/V_I^\gamma$$

and

$$p'_{II} = (p_I V_I^\gamma)/V_{II}^\gamma.$$

Introducing these expressions into Eq. (15) and assuming that the two tubes are neighboring ones so that $p_{II} = p_I + \Delta p$ and $V_{II} = V_I + \Delta V$ we obtain

$$\delta W = \Delta V[\Delta p + (\gamma p/V)\,\Delta V], \tag{16}$$

in which the subscript I has been omitted and the terms higher than second order with respect to Δp and ΔV have been neglected. Therefore, if

$$\Delta V[\Delta p + (\gamma p/V)\,\Delta V] < 0, \tag{17}$$

the system is unstable with respect to such an interchange. Grad[34] proved:

$$\Delta V[\Delta p + (\gamma p/V)\,\Delta V] > 0 \tag{18}$$

is necessary and sufficient for the stability. Since $V = AL = \phi L/B$, and ϕ is constant, $\Delta V = \phi\,\Delta(L/B)$. Introducing U through the equation[38]

$$U \equiv -\int \frac{dl}{B},$$

with dl the line element along the tube, we have

$$V = -\phi U = \phi|U| \quad \text{and} \quad \Delta V = -\phi\, \Delta U.$$

Hence, in terms of U, condition (17) takes the form

$$\Delta U\, \Delta p > \gamma p\, (\Delta U)^2/|U|. \tag{I}$$

Namely, if this condition is valid then the equilibrium is unstable.

Let us now consider the boundary condition of type II. At the boundary of the plasma p becomes zero, and so condition (17) then reduces to

$$\Delta U\, \Delta p > 0$$

or to

$$\frac{\Delta U}{\Delta \phi}\frac{\Delta p}{\Delta \phi} > 0.$$

In an axially symmetric system, p tends to zero as ϕ increases and hence $\Delta p/\Delta\phi < 0$. Therefore, if $\Delta U/\Delta\phi < 0$ or, equivalently, if

$$\delta \int \frac{dl}{B} > 0 \tag{19}$$

(that is to say if $\int dl/B$ increases outward), the system is unstable. Hence if B decreases toward the boundary and the magnetic lines of force simultaneously lengthen, the system is unstable, as might be the case for an entirely convex boundary. Since under the present conditions the magnetic field at the boundary may be assumed to be identical with the vacuum field, condition (19) may be written in the form

$$\int \frac{dl}{RrB^2} < 0, \tag{20}$$

in which R is the radius of curvature of the line of force under consideration and r is the distance of the line from the axis of symmetry. The radius of curvature R is positive if the center of curvature lies outside the plasma. The derivation of this condition has been given by Rosenbluth and Longmire.[20a]

If the plasma boundary is everywhere convex so that $R < 0$, condition (20) is obviously valid [see Eq. (13)]. In the magnetic mirror configuration shown in Fig. 10 the boundary consists of both the convex and the concave portions, and it is necessary to integrate over the line of force in order to verify condition (20). The result of the calculation shows that the negative contribution dominates the positive one, resulting in the inequality

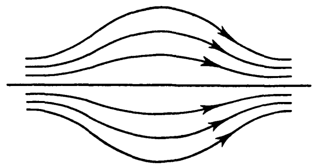

FIG. 10. Magnetic mirror.

$\int dl/RrB^2 < 0$, and therefore the system is unstable. A full discussion of this has been given by Rosenbluth and Longmire[20a] and by Thompson.[33]

The order of magnitude of the growth rate of the instability, $|\omega|$, may be estimated as

$$|\omega| \approx \sqrt{g_{\text{eff}} k}, \tag{21}$$

where g_{eff} is given by Eq. (14).[33a] Assuming that $p_\perp \approx p_\parallel \approx \frac{1}{2}\rho v_{\text{th}}^2$, where v_{th} is the thermal velocity of the ions, we find that

$$|\omega| \approx v_{\text{th}}(m/Rr_0)^{1/2}, \tag{22}$$

in which we have used the approximation $k \approx m/r_0$, where r_0 is the mean radius of the mirror and m is an integer characterizing the azimuthal mode. This instability is stabilized by the effect of conducting end plates in just the same way as was the gravitational instability. Inserting the above g_{eff} into Eq. (8c) and putting $k_z^{-1} = L$ (the length of the mirror) gives the result

$$\frac{p}{B^2} < \frac{r_0 R}{mL^2}, \tag{23}$$

which exhibits the properties of the exact calculation performed by Berkowitz, Grad, and Rubin[30,34] showing stabilization for a sufficiently low pressure.

For sufficiently large $R \approx r_0^3/r_{iL}^2$ the growth rate is small, and we have $|\omega|/\omega_{ic} \approx (r_{iL}/r_0)^2$ and hence stabilization due to finite ion Larmor radius may be expected. The stability condition obtained by Rosenbluth and others[25a] is

$$\frac{\sqrt{m}}{m-1}\sqrt{\frac{r_0}{R}} < \frac{r_{iL}}{r_0},$$

[33] W. B. Thompson, "Plasma Physics," Chap. 6, Section 2. Macmillan (Pergamon), New York, 1963.

[33a] All the values in the expression should be considered as the averages taken over lines of force.

[34] H. Grad, Some new variational properties of hydromagnetic equilibria. *Phys. Fluids* **7**, 1283–1292 (1964).

where the $m = 1$ mode is not stabilized. This is due to the fact that the electric field is constant for this mode and, consequently, the ions and electrons both experience the same electric field. It was found recently by Ioffe and Yushmanov[35] that superimposing the mirror and a cusped field produces a remarkable stabilizing effect. The cusped field was in fact produced by placing wires along the lines of force of the mirror. This stabilizing effect seems to be due to the occurrence of a minimum of the magnetic field strength $|\mathbf{B}|$. In an ordinary mirror the magnetic field increases along the magnetic lines of force and decreases radially outward, but, when a cusped field is superimposed, at the minimum point the resulting field increases in both the radial and the axial directions.

As has already been explained, the surface of a plasma in which the pressure is isotropic must be a magnetic surface generated by lines of force. However, a magnetic isobar, a surface of constant $|\mathbf{B}|$, will, in general, cut lines of force, as is the case at the ends of the mirror, so that it cannot be a magnetic surface. Hence, for the confinement of a plasma by means of a magnetic isobar, an anisotropic pressure is essential. The theory of plasma confinement and stability in a field with minimum $|\mathbf{B}|$ has been discussed by Taylor[35a] on the basis of the CGL approximation. He then derived the results more generally for adiabatic mirror machines in terms of an equilibrium distribution function.[36,36a] Recently Furth and Rosenbluth[37] have shown by example that, for closed configurations with an isotropic pressure p, the condition $\nabla p \cdot \nabla B < 0$ can be met in the average sense, that is,

$$\nabla p \cdot \nabla \int \frac{dl}{B} > 0.$$

So far we have considered the cases in which the right-hand side of the inequality (I) has been neglected. However Kadomtsev and others[38–40]

[35] M. S. Ioffe and E. E. Yushmanov, Experimental investigation of plasma instability in a magnetic mirror trap. *Nucl. Fusion: Suppl.* No. 1, 177–182 (1962).

[35a] J. B. Taylor, *Phys. Fluids* **6**, 1529–1536 (1963). This paper is included in the present volume (p. 215).

[36] J. B. Taylor, Equilibrium and stability of plasma in arbitrary mirror fields. *Phys. Fluids* **7**, 767–773 (1964). This paper presents a rather more general derivation of the results contained in the paper by Taylor.[35a]

[36a] R. J. Hastie and J. B. Taylor, Maximum plasma pressure for stability in magnetic fields with finite minima. *Phys. Fluids* **8**, 323–331 (1965).

[37] H. P. Furth and M. N. Rosenbluth, Closed magnetic vacuum configurations with periodic multipole stabilizations. *Phys. Fluids* **7**, 764–766 (1964).

[38] B. B. Kadomtsev, Plasma physics and the problem of controlled thermonuclear reactions. *Progr. Nucl. Energy, Ser. IV* **3**, 17–25 (1960); Magnetic traps for plasmas. *Progr. Nucl. Energy, Ser. IV* **3**, 417–430 (1960).

have shown some stable configurations in which p and U can be varied sufficiently to ensure that the inequality $\Delta U\,\Delta p + \gamma p(\Delta U)^2/U < 0$ is valid. For example, in an axially symmetric system let $dp/dr < 0$ and $dU/dr < 0$, then the stability condition reduces to

$$\frac{dp/dr}{\gamma p} + \frac{dU/dr}{U} > 0. \tag{I$'$}$$

Thus the system is stable with respect to interchange provided p does not decrease faster than $|U|^{-\gamma}$ with increasing r. The situation is closely analogous to that of thermal convection produced by an inhomogeneous temperature distribution when subject to the influence of gravity. As may be deduced from the convection process in a fluid heated from below, if the temperature increases vertically upward then convection does not occur and the fluid is stable. This is, however, only a sufficient condition. The necessary condition may be stated as the following requirement: As an element at height z moves upward to a new position $z + \xi$, the resultant force acts downward, and consequently the mass density there must increase or, equivalently, the specific volume V must decrease giving

$$V(p', S') - V(p', S) > 0,$$

where p' is the pressure at $z + \xi$ and S' and S are the entropies at $z + \xi$ and z, respectively. From this inequality it follows that[41]

$$\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial T}{\partial z} + \frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p\right) > 0,$$

where the suffix p signifies that pressure remains constant.

Since most substances expand on heating, and so $(\partial V/\partial T)_p > 0$, the stability condition becomes

$$\frac{\partial T}{\partial z} + \frac{gT}{c_p V}\left(\frac{\partial V}{\partial T}\right)_p > 0. \tag{I$''$}$$

Hence, even if $\partial T/\partial z < 0$, the system is stable provided the temperature decrease satisfies the above inequality. We find here a close similarity

[39] S. I. Braginskij and B. B. Kadomtsev, Stabilization of a plasma by the use of guard conductors. *Progr. Nucl. Energy, Ser. III* **3**, 356–385 (1960).

[40] B. B. Kadomtsev and S. I. Braginskij, Stabilization of a plasma by non-uniform magnetic fields. *Proc. U.N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958* Vol. 32, pp. 233–238. Columbia Univ. Press (I. D. S.), New York, 1959.

[41] See L. D. Landau and E. M. Lifshitz, "Fluid Mechanics," §4. Macmillan (Pergamon), New York, 1959.

between the inequalities (I′) and (I″). In this sense this interchange instability is often called the *convection instability*.

Let us now proceed to a discussion of the instabilities occurring in the pinched discharge. Suppose that a current flows along a surface of a cylindrical plasma column so that an azimuthal magnetic field is produced in the vacuum surrounding the plasma surface. Let us first assume that the magnetic field does not exist inside the plasma. The field and the current produce an inward Lorentz force upon the plasma which is balanced by the mechanical pressure difference across the plasma surface. The equilibrium thus obtained is not stable since the cylindrical column is obviously convex to the vacuum so that the center of curvature of the lines of force lies inside the plasma. In fact when we consider a radial perturbation normal to the magnetic field the discussion closely parallels that for the gravitational-type instability. Figure 11 illustrates a vertical section of the plasma column and the surrounding magnetic lines of force. The plasma

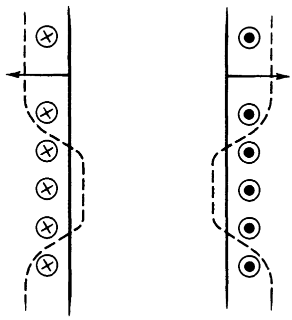

FIG. 11.

boundary is shown as the two solid lines. Owing to the curvature of the lines of force the effective force is directed outward from the plasma. Hence the perturbation shown by the dotted lines leads to an interchange instability (see Fig. 1). The growth rate is given by Eq. (14) and for an isotropic pressure, $p_{\parallel} = p_{\perp} = p$, it becomes $\sqrt{2pk/\rho R}$. Since, to within an order of magnitude, $k \approx 1/R \approx 1/r_0$, where r_0 is the radius of the column, we find that

$$|\omega| \approx \sqrt{\frac{2p}{\rho}} \Big/ r_0 \approx a/r_0,$$

where a is the sound velocity for a compressed plasma. Namely, the growth rate is roughly equal to the inverse of the transit time of a sound wave across the radius of the column. Since for a high temperature pinched plasma r_0 is usually small and a is large, $|\omega|$ is large. In other words, unlike the gravitational and mirror instabilities, an interchange instability in a pinched discharge has a large growth rate. As a result, when r_0 is sufficiently small, the plasma column is broken by the fast inward motion of the instability. In this sense such an instability is often called a *sausage-type* instability.

The stabilization of a sausage-type instability may be achieved by applying an internal shear field in just the same way as in the gravitational instability. Suppose that inside the plasma a uniform magnetic field B_z is superimposed along the column axis. Then, due to the perturbation illustrated in Fig. 11, an Alfvên wave propagates along the applied uniform field and, from Eq. (8c′), we have

$$\omega^2 = -\left(\frac{2p}{\rho}\right) \Big/ r_0^2 + \left(\frac{b}{r_0}\right)^2,$$

where b is the Alfvên speed and is equal to $\sqrt{B_z^2/4\pi\rho}$. Hence, by means of the condition

$$p + \frac{B_z^2}{8\pi} = \frac{B_\theta^2}{8\pi},$$

where B_θ is the vacuum azimuthal field, the stability condition is given by

$$B_z^2 > \tfrac{1}{2}B_\theta^2.$$

However this shear stabilization is not effective for long wavelength perturbations of the *kink type*, which are associated with the displacement of the column as a whole.

Suppose that the plasma column is bent slightly, as shown in Fig. 12. The density of lines of force surrounding the column then becomes higher on the concave side than on the convex side of the column, and, consequently, the azimuthal vacuum field exerts a stronger magnetic pressure

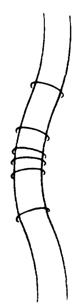

FIG. 12. Kink-type instability.

on the concave side thereby increasing the curvature. On the other hand the bending of the internal uniform magnetic field produces a magnetic tension which acts as a restoring force which competes with the destabilization produced by the azimuthal magnetic field. The result of computations shows that for long wavelength bending the former effect dominates the latter and that the system is unstable. In order to estimate these forces we first note that the Lorentz force can be expressed as the divergence of the magnetic part of the Maxwell stress tensor,

$$\mathbf{T}_{ik}^{(m)} = \frac{1}{4\pi}(B_iB_k - \tfrac{1}{2}B^2\delta_{ik}).$$

Hence the volume force $\mathbf{K}$ resulting from the Lorentz force which acts on the whole plasma column takes the form

$$\mathbf{K} = \int \operatorname{div} \mathbf{T}^{(m)}\, d\tau = \int \mathbf{T}^{(m)} \cdot \mathbf{n}\, d\sigma,$$

where the integration is extended over an arbitrary closed volume in the vacuum enclosing the plasma column, $\mathbf{n}$ is the unit normal to the surface of the volume directed outward, and $d\sigma$ is a surface element. By means of this equation the force acting upon the boundary due to the change of the vacuum magnetic pressure can be estimated as follows.[12]

Let the wavelength of the bending be λ. Then the resulting change of the vacuum magnetic field would be appreciable in a region within a distance λ of the plasma boundary.[41a] Hence it is convenient to choose the domain of integration as the cylinder of radius λ surrounding the column, so that the contribution of the perturbed field over the cylindrical boundary of the domain may be neglected as being small. (The contribution of the unperturbed field balances the unperturbed total pressure inside the plasma.) The upper and lower boundaries of the domain of integration may be specified as being the two planes separated by a distance λ and passing through the center of curvature of bending (cf. Fig. 13). The force resulting from the perturbed vacuum field is then given by the integral $\int \mathbf{T}^{(m)} \cdot \mathbf{n}\, d\sigma$ over the two end planes outside the cross sections of the column. For the component of force perpendicular to the axis of the column the integrands reduce locally to

$$\mathbf{T}_{xx}^{(m)} n_x = -(B_\theta{}^2/8\pi)(\lambda/2R),$$

where R is the radius of curvature of bending and the x axis is taken perpendicular to the column axis and is directed positively toward the center

[41a] Note that the solution of $\nabla^2\varphi = 0$ is given by $d^2\varphi_k/dx^2 - k^2\varphi_k = 0$ for perturbations in the y direction, i.e., for $\varphi \sim \varphi_k e^{iky}$.

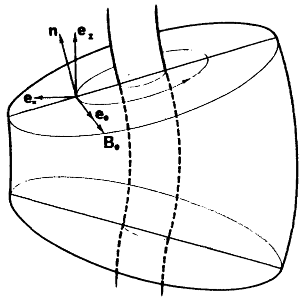

FIG. 13.

of curvature; the expression being the same on either plane. (Since B_θ is orthogonal to the x axis, $T_{xy}^{(m)}$ and $T_{xz}^{(m)}$ are zero.) Hence the integral may be approximated by

$$\frac{-\lambda}{4R}\int_{r_0}^{\lambda} B_\theta^2 r\,dr \approx -\frac{\lambda}{4R} r_0^2 B_{\theta_0}^2 \ln(\lambda/r_0),$$

where B_{θ_0} is the value of B_θ on the boundary. This equation indicates that the perturbation of the vacuum field induces a force in a direction opposite to that of the curvature, that is, directed toward the convex side. The strength of this force per unit length becomes

$$\left(\frac{r_0^2}{4R}\right) B_{\theta_0}^2 \ln(\lambda/r_0).$$

The restoring force resulting from the B_z field inside the plasma is given by integrating over the cross sections of the column on each end plane. Noting that by means of the frozen-in condition the internal magnetic lines of force are always aligned with **n** we find, in this case, that the integrand is positive, and so the resultant force acts to restore the displacement. Instead we may, of course, estimate the force as the magnetic tension $(1/4\pi)B_z^2/R$ when we find that the total magnetic tension acting upon the column is $(B_z^2/4R)r_0^2$ per unit length. As a result we obtain the stability condition

$$B_z^2 - B_{\theta_0}^2 \ln(\lambda/r_0) \gtrsim 0$$

or

$$(B_z/B_{\theta_0})^2 \gtrsim \ln(\lambda/r_0).$$

However the condition

$$p + B_z^2/8\pi = B_{\theta_0}^2/8\pi$$

implies that $B_{\theta_0} > B_z$, and so the internal shear field B_z cannot stabilize the pinch against long wavelength kinks.

To stabilize a kink instability we must consider the following possibilities:

(1) Superposition of an external axial magnetic field
(2) Placing a coaxial conducting wall around the vacuum field.

The first method is not effective however due to the fact that there exists a perturbation mode having the same pitch as the external magnetic field so introduced. This is given by $2\pi r B_z/B_\theta$ or, in terms of the axial current I that can be introduced through the equation $B_\theta = 2I/cr$, it can be written as $c\pi r^2 B_z/I$. Since the pitch of the disturbance produced by the perturbation cannot exceed the length L of the pinch itself, the stability condition becomes a condition for the current I; that is $L < c\pi r^2 B_z/I$ or $I < c\pi r^2 B_z/L$. Namely, if the discharge current I exceeds the critical value $I_c = c\pi r^2 B_z/L$, then the instability will appear. This result was found independently by Kruskal and Shrafranov and has since received experimental confirmation.[41b,42,43] The critical current I_c is often called the Kruskal-Shrafranov limit. The kink may however be stabilized by utilizing the effect produced by a conducting wall. This can be easily understood by noting that the magnetic pressure increases on the convex side as it approaches the wall. However, the radius of the conducting wall must be kept less than five times the radius of the column as was shown by Taylor[43a] and others.[42–44]

The problem of a stabilized pinch has been discussed by many authors, and it is not easy to give a complete survey. Instead, we shall now briefly introduce the relevant papers included in this edition and mention some of their principal references. Kruskal and Schwarzschild[12a] discussed the kink instability corresponding to $m = 1$ in the azimuthal eigenmode (m), in the absence of internal and external axial fields and of a conducting wall. Kruskal and Tuck[44] have taken into account the effect of both an internal

[41b] M. D. Kruskal, J. L. Johnson, M. B. Gottlieb, and L. M. Goldman, *Phys. Fluids* **1,** 421–429 (1958). This paper is included in the present volume (p. 139).

[42] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Some stable hydromagnetic equilibria. *Phys. Fluids* **1,** 281–296 (1958).

[43] G. G. Dolgov-Saveliev, V. S. Mukhovatov, V. S. Strelkov, N. N. Shepelev, and N. A. Yavlinskii, Investigation of a toroidal discharge in a strong magnetic field. *Proc. Intern. Conf. Ionization Phenomena Gases, 4th, Uppsala, 1959* Vol. II, pp. 947–953. North-Holland Publ., Amsterdam, 1960.

[43a] R. J. Tayler, *Proc. Phys. Soc. (London)* **B70,** 1049–1063 (1957a). This paper is included in the present volume (p. 149).

[44] M. D. Kruskal and J. Tuck, Instability of a pinched fluid with a longitudinal magnetic field. *Proc. Roy. Soc. (London)* **A245,** 222–237 (1958).

and an external axial field and have investigated all possible modes of perturbation [$m = 0$ (the sausage), $m = 1$, and $m \geq 2$]. However, the effect of a wall was not considered. The detailed stability condition for a cylindrical pinch with a coaxial conducting wall was obtained by Taylor,[43a] Rosenbluth,[45] Shrafranov,[46] and Chandrasekhar and others,[47] who gave conditions expressed in terms of the ratio of the axial field strength to the azimuthal field strength and the ratio between the radius of the pinch and the coaxial wall as parameters. The conditions are applicable to a pinch having an infinitely thin surface layer of current. It was pointed out by Rosenbluth[47a] that taking into account the thin but finite width of the current layer results in a new type of instability called the *surface instability.*

Inside the current layer the magnetic field changes continuously so that the lines of force may be described in terms of a set of spirals with a pitch

$$\bar{\mu} = B_\theta / rB_z,$$

which varies continuously as a function of radial distance. On a certain cylindrical surface in the current layer consider a spiral displacement of the plasma having the same pitch as the lines of force. The lines of force would not be bent by such a displacement, and so we have found yet another perturbation which is likely to produce instability. In such a case the stability condition becomes more complicated since it then depends on the structure of the magnetic field in the current layer, and no stable layer exists if the external field is in the same direction as the internal field. The surface on which the matching of the pitches occurs corresponds to a singular point in the Euler-Lagrange equation of the variation integral, and the region interior to the singular surface becomes separated from the region exterior to the surface. As a result different stability conditions must be satisfied in each region. In general the stability condition in the interior region may be satisfied more easily than the stability condition in the exterior region, and so it appears that the system is stable in the interior region and that the instability is confined to the exterior region. The growth rate is small and is proportional to the width of the surface layer, and it appears that a finite Larmor radius effect could modify the result.[25a]

[45] M. N. Rosenbluth, Stability of the pinch. *Los Alamos Rept. LA2030,* 1956.

[46] V. D. Shafranov, On the stability of a cylindrical gaseous conductor in a magnetic field. *J. Nucl. Energy II* **5,** 86–91 (1957).

[47] S. Chandrasekhar, A. N. Kaufman, and K. N. Watson, The stability of the pinch. *Proc. Roy. Soc.* (*London*) **A245,** 435–455 (1958).

[47a] M. N. Rosenbluth, *Proc. U.N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958* Vol. 31, pp. 85–91. Columbia Univ. Press (I.D.S.), New York, 1959. This paper is included in the present volume (p. 205).

From the discussion of the surface layer instability it may be expected that a similar instability occurs for a thick current layer. The necessary condition obtained by Suydam[47b] for the $m \neq 0$ mode of a cylindrical pinch to be stable is that

$$(r/4)(\bar{\mu}'/\bar{\mu})^2 + 8\pi p'/B_z \geq 0$$

at every point in the plasma. Here the prime denotes differentiation with respect to the radial coordinate r. This condition implies that if the pressure varies steeply then a greater spiral pitch is required. The necessary and the sufficient condition has been given by Newcomb[47c] who in turn derived the Rosenbluth condition for a thin surface layer of current. Though the necessary and sufficient condition for the stability of a cylindrical pinch is not intuitively obvious, the variational approach leads to the following simple form of the sufficient condition:

$$\frac{B_\theta}{r}\frac{d}{dr}(rB_\theta) < 0.$$

Noting that

$$\frac{1}{r}\frac{d(rB_\theta)}{dr} = 4\pi j_z,$$

and integrating this equation, we find that the above inequality reduces to

$$\frac{I_z}{r} j_z < 0,$$

where I_z is the total current and is equal to $\int 4\pi r^2 j_z \, dr$. Namely, if the current density in the plasma is opposite in sign to the total current then the system is stable. Such a configuration is realized in the so-called hard core pinch, in which a current I_0 flows through a central core and returns through a hollow shell of plasma. Hence, the direction of the current density flowing through the plasma at a distance r from the center of the core is opposite to that of the current flowing through the core itself.[33,47c] On the other hand, the total current passing through the circular surface of radius r centered on the core is equal to the sum of the current densities flowing through the hollow plasma inside the circle and the current I_0 flowing through the core itself. Therefore the sign of this total current can be made opposite to that of the current density at a point inside the circle if, for example, $|I_0|$ is sufficiently large.

[47b] B. R. Suydam, *Proc. U.N. Intern. Conf. Peaceful Uses At. Energy, 2nd, Geneva, 1958* Vol. 31, pp. 157–159. Columbia Univ. Press (I.D.S.), New York, 1959. This paper is included in the present volume (p. 165).

[47c] W. L. Newcomb, *Ann. Phys.* (*N.Y.*) **10,** 232–267 (1960). This paper is included in the present volume (p. 169).

For the limit in which the plasma occupies a thin tubular layer the necessary and sufficient conditions for stability have been obtained in a much simplified form by Newcomb and Kaufman.[48] By specifying some further parameters in the results they also discussed the necessary and sufficient conditions for the stability of a columnar pinch and compared the results with the tubular pinch.

In spite of this absolute stability of the hard core pinch a complete stabilization has not been observed, and it is possible that effects such as resistivity which cannot be taken into account in ideal magnetohydrodynamics may give rise to further instabilities.

In fact in the limit of very small conductivity it has already been shown that instabilities may develop due to the rapid penetration of the magnetic field into the plasma. The theory is in good agreement with an experiment using liquid mercury when it is suitably modified to include the effect of surface tension.[49] Assuming that the conductivity is very large within a thin boundary layer beyond which the plasma is of low conductivity (thereby being completely decoupled from the magnetic field), Jukes[14] has also shown theoretically that a slip instability is possible. It is interesting to note that in his results the unstable mode is given by the condition $\mathbf{k} \times \mathbf{B} = 0$.

A most comprehensive discussion for high conductivity plasmas can be undertaken on the basis of the results of Furth *et al.*[13a] First of all we note the tearing mode which results from the structure of the sheared magnetic field. As may be seen from Fig. 5, this mode is given by

$$\mathbf{k} \cdot \mathbf{B} = 0$$

which, in the superposed axial and azimuthal magnetic fields, takes the well-known form

$$\frac{m}{r} B_\theta + k B_z = 0.$$

Since the tearing mode is suppressed for short wavelength perturbations, it can be stabilized if B_θ / B_z is sufficiently large. A similar result has been obtained independently by Rebut.[50] Instead of explicitly taking into account the effects of finite conductivity, he first finds a helical buckling equilibrium neighboring a cylindrical equilibrium. He then deduces that whenever

[48] W. A. Newcomb and A. N. Kaufman, Hydromagnetic stability of a tubular pinch. *Phys. Fluids* **4,** 314–334 (1961).

[49] R. J. Bickerton and I. J. Spalding, The hydromagnetic stability of the hard core pinch with small electrical conductivity. *J. Nucl. Energy: Pt. C* **4,** 151–158 (1962).

[50] P. H. Rebut, Non-magnetohydrodynamic instabilities in plasmas of high current density. *J. Nucl. Energy: Pt. C* **4,** 159–168 (1962).

these exist the discharge is unstable. However, in the framework of ideal magnetohydrodynamics, the displacements necessary to attain these equilibria involve discontinuous changes of the magnetic field such as cutting and rejoining lines of force, and they only become possible for a plasma if we allow finite conductivity effects. This may also be seen from the work by Furth *et al.*[13a] Namely, outside the small region where the motion of the plasma and the magnetic field are decoupled, we must seek to find those ideal magnetohydrodynamic solutions which cannot be joined without first taking into account finite conductivity effects.

The explicit statement for the hard core pinch is as follows. The pinch will be stable if

$$B_\theta/B_z > R/\delta;$$

that is if $I_{\text{hard core}}/I_{\text{plasma}} > R/\delta$, where $I_{\text{hard core}}$ and I_{plasma} are the currents in the central rod and the plasma, respectively, R is the radius of the discharge channel, and δ is its thickness. If this condition is not satisfied then the unstable mode mentioned above becomes possible at the radius at which $dp/dr = 0$.

The instability observed in experiments[51] corresponds to the predicted tearing mode while the growth rate is also in agreement with the theoretical prediction by Fürth *et al.*

The rippling mode due to the spatial variation of electrical conductivity is also possible, but for high temperature plasmas it is likely that this may be stabilized by the effect of thermal conduction.

The systems which have so far been discussed are open-ended, and for the real containment of a plasma we are led to consider toroidal configurations. As has already been remarked,[10,43] the simplest toroidal configuration in which the lines of force close themselves along a long path around the torus cannot be an equilibrium configuration. This may be seen very simply by noting that the drifts produced by the curvature of the lines of force result in charge separation thereby causing the plasma to be pushed to the walls. However this charge separation is canceled out by twisting the lines of force in such a manner that they close themselves after many turns around the torus.

Let a line of force start at a point P_1 on a cross section of the torus and, after traversing a long path around the torus, let it return to another point P_2 on the cross section. On the cross section we thus have successive rotational transforms which cause all such points to lie close to a single closed

[51] K. L. Aitken, R. J. Bickerton, P. Ginot, R. A. Hardcastle, A. Malein, and P. Reynolds, The stability of linear pinch and hard core discharges. *J. Nucl. Energy: Pt. C* **6,** 39–69 (1964).

curve. As a result, after many turns around the torus a single line of force will generate a magnetic surface the center line of which is called the magnetic axis. It should be noted that, as in the case of Tokamak,[52,53] the rotational transform can be realized even in a simple torus provided a plasma current flows along the magnetic field. It is well known that the rotational transform is most simply produced by twisting a torus into a figure eight in the manner proposed by Spitzer.[10] However in this case the rotational transform is constant on the cross section, and so the interchange instability is possible. To overcome this instability the rotational transform may be varied with the radial distance from the magnetic axis so that the magnetic field has a shear in the radial direction.[11] It should however be emphasized that under this shear stabilization a kink-type instability[53a] will appear as the current exceeds the Kruskal limit which is then given by a slight modification of the expression already given for a cylinder.[11,41b,42,43,54] The general analysis of the necessary and sufficient stability conditions for the toroidal configuration is complicated, but an extensive study has nevertheless been made by several authors.[55–59]

Finally we note that for a collisionless plasma a new type of instability associated with an anisotropic pressure can be predicted. Namely the Alfvén wave and the slow compressive hydromagnetic wave becoming unstable. The former instability occurs when $p_{\|} \gg p_{\perp}$ and has been called the *fire-hose instability*,[13,60] while the latter is called the mirror instability and takes place for sufficiently large $p_{\perp}$.[13]

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An energy principle for hydromagnetic stability problems

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The problem of the stability of static, highly conducting, fully ionized plasmas is investigated by means of an energy principle developed from one introduced by Lundquist. The derivation of the principle and the conditions under which it applies are given. The method is applied to find complete stability criteria for two types of equilibrium situations. The first concerns plasmas which are completely separated from the magnetic field by an interface. The second is the general axisymmetric system.

1. Introduction

The investigation of hydromagnetic systems and their stability is of interest in such varied fields as the study of sunspots, interstellar matter, terrestrial magnetism, auroras and gas discharges. An excellent summary and bibliography of these applications has been given by Elsasser (1955, 1956). The stability of hydromagnetic systems has been extensively investigated in a fundamental series of papers by Chandrasekhar (1952 to 1956).

The present work is concerned with those hydromagnetic equilibria in which the fluid velocity at each point is assumed to vanish. It is divided into two parts. The first is a development of an energy principle, originally stated by Lundquist (1951, 1952), for investigating the stability of such systems. The second part consists of the application of this principle to obtain a number of specific results for such systems.

The 'normal mode' technique is the usual method for the investigation of stability in many systems, mechanical, electrical, etc. It consists of solving the linearized equations of motion for small perturbations about an equilibrium state. The system is said to be unstable if any solution increases indefinitely in time; if no such solution exists, the system is stable.

The energy principle technique, on the other hand, depends upon a variational formulation of the equations of motion. It was first used by Rayleigh (1877) in the calculation of the frequencies of vibrating systems. Its advantage lies in the fact that if one seeks solely to determine stability, and not rates of growth or oscillation frequencies, it is necessary only to discover whether there is any perturbation which decreases the potential energy from its equilibrium value. This makes practical the stability analysis of much more complicated equilibria than the normal mode method.

In § 2 are presented the basic equations for a plasma and the conditions under which they are valid. These equations are then linearized in the Lagrangian representation. In § 3, the energy principle is stated and derived from the normal mode equations for the system. The relation between the energy principle and Rayleigh's principle (Rayleigh 1877) is discussed.

In § 4, some convenient methods for applying the energy principle to general problems are described. In § 5, the problem of the stability of a fluid in which the magnetic field is zero and which is surrounded by a vacuum magnetic field is solved.

Section 6 treats the stability of a general axisymmetric system. The problem is reduced essentially to the solution of an ordinary second-order eigenvalue equation. In certain limiting situations the problem is solved completely.

2. Basic considerations

Consider a plasma of electrons and of one kind of positive ion which is governed by the following system of equations:

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \mathbf{j} \times \mathbf{B} - \rho \operatorname{grad} \phi, \tag{2·1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{2·2}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \tag{2·3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}.\operatorname{grad}\right)(p\rho^{-\gamma}) = 0, \tag{2·4}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2·5}$$

$$\operatorname{curl} \mathbf{B} = \mathbf{j}, \tag{2·6}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{2·7}$$

Let $\mathbf{E}$ be the electric field, $\mathbf{B}$ the magnetic field, $\mathbf{j}$ the electric current density, ρ the mass density, M the ion mass, p the pressure, ϕ the external potential energy per unit mass, γ the ratio of specific heats, e the magnitude of the electronic charge and $\mathbf{v}$ the fluid velocity. The equations are written in rationalized Gaussian units with $c = 1$.

The above equations apply if the following conditions are satisfied: (i) Quadratic terms in $\mathbf{v}$ and $\mathbf{j}$ are negligible. Physically, this is equivalent to the requirement that the macroscopic speed v is small compared to sound speed $c_s = (\gamma p/\rho)^{\frac{1}{2}}$ or to hydromagnetic speed $c_k = B/\sqrt{\rho}$. (ii) The system is locally electrically quasi-neutral. This occurs if the Debye shielding distance $\lambda_D = (kT_e/ne^2)^{\frac{1}{2}}$ is small compared to every characteristic dimension L of the system. (iii) The ratio of the electron mass, m to the ion mass, M is negligible compared to unity. (iv) The matter stress tensor is isotropic. This occurs if there are many collisions during a characteristic time, t_c. The effect of relaxing the requirement of isotropy of the stress tensor is considered in § 3. (v) The displacement current is negligible. This holds if c_k is small compared to the speed of light. (vi) Heat flow by conduction, along the lines of force as well as across the lines, is negligible. This implies the adiabatic law (2·4). It is shown in § 3 how this law must be modified if conditions (iv) and (vii) are not satisfied. (vii) Ohm's law in the form of equation (2·3) is valid.

Spitzer (1956) gives the complete generalized Ohm's law which may be written in the form

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{M}{e} \operatorname{grad} \phi - \eta \mathbf{j} - \frac{m}{ne^2} \frac{\partial \mathbf{j}}{\partial t} - \frac{1}{ne} \operatorname{grad} p_i - \frac{M}{e} \frac{\partial \mathbf{v}}{\partial t} = 0.$$

The electron inertia term $(m/ne^2)\,\partial\mathbf{j}/\partial t$ is negligible when $(t_c)^{-1}$ is small compared to the electron plasma frequency $\omega_p = (ne^2/m)^{\frac{1}{2}}$. The ion inertia term $(M/e)\,\partial\mathbf{v}/\partial t$ is negligible when $(t_c)^{-1}$ is small compared to the ion Larmor frequency eB/M. The electrical resistance term $\eta\mathbf{j}$ is negligible when the time characteristic of relative diffusion of matter and magnetic flux is long compared to t_c. The terms involving $\operatorname{grad}\phi$ and $\operatorname{grad}p_i$ are negligible when $a_i c_s/Lv \ll 1$, where a_i is the ion Larmor radius. Spitzer has pointed out that this criterion is not satisfied in general for fully ionized plasmas. In particular, for equilibrium states in which v is zero, the criterion fails. The effect of keeping these terms is discussed in § 3 where it is shown that the stability criteria are not affected by their inclusion.

The set of equations above implies relations between quantities on adjacent sides of an interface, either interior to the fluid or between fluid and vacuum. Denote by $\mathbf{n}$ the unit normal to the interface, by $\mathbf{K}$ the surface current density, and by $\langle X\rangle$ the increment in any quantity X across the boundary in the direction $\mathbf{n}$. For a fluid-fluid interface the relations are

$$\langle p+\tfrac{1}{2}B^2\rangle = 0, \tag{2·8}$$

$$\mathbf{n}.\langle\mathbf{v}\rangle = 0, \tag{2·9}$$

$$\mathbf{n}\times\langle\mathbf{E}\rangle = \mathbf{n}.\mathbf{v}\langle\mathbf{B}\rangle, \tag{2·10}$$

$$\mathbf{n}.\langle\mathbf{B}\rangle = 0, \tag{2·11}$$

$$\mathbf{n}\times\langle\mathbf{B}\rangle = \mathbf{K}. \tag{2·12}$$

For a fluid-vacuum interface equation (2·9) is meaningless, but the remaining relations apply with $\mathbf{v}$ taken to be the fluid velocity.

The region of interest can often be considered surrounded by a rigid, perfectly conducting wall. At such a boundary the appropriate conditions are

$$\mathbf{n}\times\mathbf{E} = 0, \tag{2·13}$$

$$\mathbf{n}.\partial\mathbf{B}/\partial t = 0, \tag{2·14}$$

$$\mathbf{n}.\mathbf{v} = 0. \tag{2·15}$$

A further condition which must be satisfied at any interface carrying a sheet current, but no sheet mass, is that the lines of force of the magnetic field lie in the interface. This arises from the fact that refraction of the lines of force would give rise to infinite accelerations in the surface due to the unbalanced tangential forces. Throughout this paper, only surfaces of discontinuity will be considered at which the condition $\mathbf{n}.\mathbf{B} = 0$ is satisfied. This is the boundary condition of interest, for example, for a confined plasma in which gravitational effects are negligible.

It can be shown that the system of equations above possess an energy integral

$$U = \int d\tau\left(\tfrac{1}{2}\rho\,|\,\mathbf{v}\,|^2+\tfrac{1}{2}\,|\,\mathbf{B}\,|^2+\frac{p}{\gamma-1}+\rho\phi\right) = \text{constant}, \tag{2·16}$$

where the integration is extended over the whole domain, fluid and vacuum.

It is convenient in later exhibiting the energy principle for the linearized form of the above equations to adopt a Lagrangian description of the fluid motion. Accordingly, all quantities are now considered to be functions of $\mathbf{r}_0$, the initial location of a fluid element, and of t, the time. Let the displacement vector $\boldsymbol{\xi}(\mathbf{r}_0, t)$ be determined by

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi}, \tag{2·17}$$

where $\mathbf{r}$ is the location of the fluid element at time t. Clearly $\boldsymbol{\xi}(\mathbf{r}_0, 0)$ is zero. Define grad_0 to be the gradient operator with respect to $\mathbf{r}_0$. The usual chain rule of differentiation yields

$$\text{grad} = \text{grad}\,\mathbf{r}_0 . \text{grad}_0. \tag{2·18}$$

To first order in $\boldsymbol{\xi}$ equation (2·18) becomes

$$\text{grad} = \text{grad}_0 - (\text{grad}_0\,\boldsymbol{\xi}) . \text{grad}_0. \tag{2·19}$$

Consider systems which are passing through a configuration of static equilibrium at time zero. The equilibrium equations are

$$\text{grad}_0\, p_0 - \mathbf{j}_0 \times \mathbf{B}_0 + \rho_0\, \text{grad}_0\, \phi_0 = 0, \tag{2·20}$$

$$\text{curl}\,\mathbf{B}_0 = \mathbf{j}_0, \tag{2·21}$$

$$\text{div}_0\,\mathbf{B}_0 = 0. \tag{2·22}$$

The equations determining the various perturbed field quantities at $\mathbf{r}$ to first order in $\boldsymbol{\xi}$ are determined by linearizing (2·1) to (2·6). There results on combining (2·3) and (2·5) and integrating in time,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{Q} + \boldsymbol{\xi} . \text{grad}_0\, \mathbf{B}_0, \tag{2·23}$$

where

$$\mathbf{Q} = \text{curl}_0\,(\boldsymbol{\xi} \times \mathbf{B}_0). \tag{2·24}$$

Equations (2·6), (2·2), (2·4) and a Taylor expansion of the external potential yield, respectively,

$$\mathbf{j} = \mathbf{j}_0 - [(\text{grad}_0\,\boldsymbol{\xi}) . \text{grad}_0] \times \mathbf{B}_0 + \text{curl}_0\mathbf{Q} + \text{curl}_0\,[(\boldsymbol{\xi} . \text{grad}_0)\, \mathbf{B}_0], \tag{2·25}$$

$$\rho = \rho_0 - \rho_0\, \text{div}_0\, \boldsymbol{\xi}, \tag{2·26}$$

$$p = p_0 - \gamma p_0\, \text{div}_0\, \boldsymbol{\xi}, \tag{2·27}$$

$$\phi = \phi_0 + \boldsymbol{\xi} . \text{grad}_0\, \phi_0. \tag{2·28}$$

The above equations are the first-order Lagrangian counterparts of (2·2) to (2·6). Note that they involve $\boldsymbol{\xi}$ but not $\dot{\boldsymbol{\xi}}$, where a dot indicates differentiation with respect to time. It can be shown that this property of depending on $\boldsymbol{\xi}$ but not $\dot{\boldsymbol{\xi}}$ holds for the expression of grad, $\mathbf{B}, \mathbf{j}, \rho, p$ and ϕ to all higher orders in $\boldsymbol{\xi}$. Finally, the equation of motion (2·1) takes the form

$$\rho_0 \ddot{\boldsymbol{\xi}} = \mathbf{F}\{\boldsymbol{\xi}\}, \tag{2·29}$$

where

$$\mathbf{F}\{\boldsymbol{\xi}\} = \text{grad}_0\,[\gamma p_0\, \text{div}_0\, \boldsymbol{\xi} + (\boldsymbol{\xi} . \text{grad}_0)\, p_0] + \mathbf{j}_0 \times \mathbf{Q} - \mathbf{B}_0 \times \text{curl}_0\, \mathbf{Q} + [\text{div}_0\,(\rho_0\, \boldsymbol{\xi})]\, \text{grad}_0\, \phi_0. \tag{2·30}$$

Note that $\mathbf{F}$ also depends only on $\boldsymbol{\xi}$ and not on $\dot{\boldsymbol{\xi}}$.

Note that (2·29) with appropriate initial and boundary conditions determines $\boldsymbol{\xi}$. Equations (2·23) to (2·28) then determines the perturbed field quantities.

The boundary conditions at an interface between a plasma and a vacuum are given by transcribing (2·8) to (2·12) to first order $\boldsymbol{\xi}$. Introduce the first-order vacuum vector potential, $\mathbf{A}$, where

$$\hat{\mathbf{E}} = -\frac{\partial \mathbf{A}}{\partial t} + \hat{\mathbf{E}}_0 \quad \text{and} \quad \hat{\mathbf{B}} = \text{curl}\,\mathbf{A} + \hat{\mathbf{B}}_0, \tag{2·31}$$

and vacuum quantities are distinguished when necessary by a circumflex. The gauge has been chosen so that the scalar potential vanishes. Then from (2·8)

$$-\gamma p_0 \operatorname{div}_0 \boldsymbol{\xi} + \mathbf{B}_0 . (\mathbf{Q} + [\boldsymbol{\xi} . \operatorname{grad}_0] \mathbf{B}_0) = \hat{\mathbf{B}}_0 . (\operatorname{curl} \mathbf{A} + [\boldsymbol{\xi} . \operatorname{grad}] \hat{\mathbf{B}}_0). \tag{2·32}$$

It follows from (2·11), (2·10) and (2·3) that

$$\mathbf{n}_0 \times \mathbf{A} = -(\mathbf{n}_0 . \boldsymbol{\xi}) \hat{\mathbf{B}}_0. \tag{2·33}$$

Of course, $\mathbf{A}$ must satisfy the equation

$$\operatorname{curl} (\operatorname{curl} \mathbf{A}) = 0 \tag{2·34}$$

in the vacuum.

Equations (2·33) and (2·34) serve to determine $\operatorname{curl}_0 \mathbf{A}$ in terms of $\boldsymbol{\xi}$, so that (2·32) is the only constraint on $\boldsymbol{\xi}$. The linearized counterpart of (2·13) which holds at a rigid, perfectly conducting wall bounding the vacuum is

$$\hat{\mathbf{n}} \times \mathbf{A} = 0. \tag{2·35}$$

At such a wall bounding a fluid, the condition is

$$\mathbf{n} . \boldsymbol{\xi} = 0. \tag{2·36}$$

3. The energy principle

On the basis of § 2, it is possible in principle to follow in time any small motion about an equilibrium state in which the fluid velocity is zero. The central problem of this paper is to determine for a given equilibrium configuration whether such a small motion grows in time. If we confine ourselves just to the question of the determination of the stability of a system and do not inquire into details of the motion, the problem may be reduced to examining the sign of the change in the potential energy as a functional of $\boldsymbol{\xi}$. It will be shown in this section that the system is unstable if, and only if, there exists some displacement $\boldsymbol{\xi}$ which makes this change in energy negative.

The demonstration demands that $\mathbf{F}$ ((2·30)) be a self-adjoint operator. That is, for any two vector fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ satisfying (2·32)

$$\int d\tau_0 \boldsymbol{\eta} . \mathbf{F}\{\boldsymbol{\xi}\} = \int d\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\boldsymbol{\eta}\}. \tag{3·1}$$

The self-adjointness property of F could be proved directly, but will be shown more simply to follow from the existence of an energy integral for the linearized system in which terms in the form of a product of $\dot{\boldsymbol{\xi}}$ and $\boldsymbol{\xi}$ do not appear. Such an energy integral for the linearized system is guaranteed in the case $\mathbf{v} = 0$ by the energy integral, (2·16), for the exact equations. In fact, the kinetic energy term for the linearized system is just

$$K\{\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\} = \frac{1}{2} \int d\tau_0 \rho_0 |\dot{\boldsymbol{\xi}}|^2, \tag{3·2}$$

while, when the potential energy terms are expanded in $\boldsymbol{\xi}$, the change in the potential energy is a quadratic form $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ which does not involve $\dot{\boldsymbol{\xi}}$ because of the remark following (2·28). Hence,

$$K\{\dot{\boldsymbol{\xi}}, \dot{\boldsymbol{\xi}}\} + \delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\} \tag{3·3}$$

is constant. One obtains from the equation of motion (2·29)

$$\dot{K} = \int d\tau_0 \dot{\boldsymbol{\xi}} . \mathbf{F}\{\boldsymbol{\xi}\} = -\delta \dot{W}$$
$$= -\delta W\{\dot{\boldsymbol{\xi}}, \boldsymbol{\xi}\} - \delta W\{\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\}. \tag{3·4}$$

Since $\dot{\boldsymbol{\xi}}$ satisfies the same boundary condition as $\boldsymbol{\xi}$, we can choose $\dot{\boldsymbol{\xi}}$ to be equal to any arbitrary displacement $\boldsymbol{\eta}$. By (3·4)

$$\int d\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\dot{\boldsymbol{\xi}}\} = \int d\tau_0 \dot{\boldsymbol{\xi}} . \mathbf{F}\{\boldsymbol{\xi}\} \tag{3·5}$$

and $\mathbf{F}$ is self-adjoint. Further the potential energy is

$$\delta W = -\frac{1}{2} \int d\tau_0 \boldsymbol{\xi} . \mathbf{F}\{\boldsymbol{\xi}\}, \tag{3·6}$$

as seen by replacing $\dot{\boldsymbol{\xi}}$ by $\boldsymbol{\xi}$ itself in (3·4).

Since the time does not appear explicitly in (2·29), one seeks normal mode solutions of the form

$$\boldsymbol{\xi}_n(\mathbf{r}_0, t) = \boldsymbol{\xi}_n(\mathbf{r}_0)\, e^{i\omega_n t}. \tag{3·7}$$

The corresponding eigenvalue equation is

$$-\omega_n^2 \rho_0 \boldsymbol{\xi}_n = \mathbf{F}\{\boldsymbol{\xi}_n\}, \tag{3·8}$$

where $\boldsymbol{\xi}_n$ satisfies the boundary condition (2·32). Since $\mathbf{F}$ is self-adjoint the eigenfunctions $\boldsymbol{\xi}_n$ can be chosen to satisfy the orthonormality condition

$$\frac{1}{2} \int d\tau_0 \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m = \delta_{nm}. \tag{3·9}$$

It is physically reasonable to assume that these eigenfunctions form a complete set for any functions which satisfy the boundary condition (2·32). (The unimportant special cases involving degeneracy of eigenfunctions will be consistently ignored.) It further follows from the fact that $\mathbf{F}$ is self-adjoint that ω_n^2 is real and thus the phenomenon of 'overstability' cannot occur.

Any eigenmode with positive ω_n^2 corresponds to a stable oscillation. A negative ω_n^2 corresponds to instability. Thus, in virtue of the assumed completeness property, the necessary and sufficient condition for instability is the existence of a negative ω_n^2.

On physical grounds one expects that if δW can be made negative then the system is unstable and therefore, there exists at least one negative ω_n^2. To show this, let $\boldsymbol{\xi}$ be a displacement which satisfies the boundary condition (2·32) and for which $\delta W < 0$. By the assumed completeness property one can write

$$\boldsymbol{\xi} = \Sigma a_n \boldsymbol{\xi}_n, \tag{3·10}$$

and from (3·6), (3·8) and (3·9)

$$\delta W = -\tfrac{1}{2} \sum_n \sum_m a_n a_m \int d\tau_0 \boldsymbol{\xi}_n . \mathbf{F}\{\boldsymbol{\xi}_m\}$$
$$= \tfrac{1}{2} \sum_n a_n^2 \omega_n^2. \tag{3·11}$$

Thus δW can be made negative if and only if there exists at least one negative ω_n^2. Therefore, the determination of the stability of a system is reduced to an examination of the sign of δW. Since the displacements $\boldsymbol{\xi}$ which may be employed in δW are subject to (2·32), the energy principle as it stands is of limited utility. It is possible to derive an extended energy principle which dispenses with this constraint.

To this end one rewrites δW as the sum of three terms, a volume integral δW_F extended over the fluid domain, a surface integral δW_S extended over the fluid-vacuum interface and a volume integral δW_V extended over the vacuum region. There results from (2·30) and (3·6) after integration by parts, suppression of the subscript zero and use of the condition $\mathbf{n}.\mathbf{B} = 0$,

$$\delta W = \delta W_F - \frac{1}{2}\int \mathrm{d}\sigma\, \mathbf{n}.\boldsymbol{\xi}[\gamma p \operatorname{div}\boldsymbol{\xi} + \boldsymbol{\xi}.\operatorname{grad} p - \mathbf{B}.\mathbf{Q}], \tag{3·12}$$

where

$$\delta W_F = \frac{1}{2}\int \mathrm{d}\tau\{|\,\mathbf{Q}\,|^2 - \mathbf{j}.\mathbf{Q}\times\boldsymbol{\xi} + \gamma p(\operatorname{div}\boldsymbol{\xi})^2 + (\operatorname{div}\boldsymbol{\xi})(\boldsymbol{\xi}.\operatorname{grad} p) - (\boldsymbol{\xi}.\operatorname{grad}\phi)\operatorname{div}(\rho\boldsymbol{\xi})\}, \tag{3·13}$$

and the integral is extended, of course, over the initial volume of the fluid. Note that the continuity of the equilibrium value of $(p + \frac{1}{2}|\,\mathbf{B}\,|^2)$ across the boundary implies the continuity of $\mathbf{n}\times\operatorname{grad}(p+\frac{1}{2}|\,\mathbf{B}\,|^2)$. This allows us with the help of equation (2·32) to rewrite the surface term in (3·12) as

$$\begin{aligned}\delta W - \delta W_F &= \frac{1}{2}\int \mathrm{d}\sigma\,\mathbf{n}.\boldsymbol{\xi}\{-\boldsymbol{\xi}.\operatorname{grad}(p+\tfrac{1}{2}|\,\mathbf{B}\,|^2) + \boldsymbol{\xi}.\operatorname{grad}(\tfrac{1}{2}|\,\hat{\mathbf{B}}\,|^2) + \hat{\mathbf{B}}.\operatorname{curl}\mathbf{A}\}\\ &= \frac{1}{2}\int \mathrm{d}\sigma\{-(\mathbf{n}.\boldsymbol{\xi})^2\,\mathbf{n}.\operatorname{grad}(p+\tfrac{1}{2}|\,\mathbf{B}\,|^2) - (\mathbf{n}.\boldsymbol{\xi})^2\,\hat{\mathbf{n}}.\operatorname{grad}(\tfrac{1}{2}|\,\hat{\mathbf{B}}\,|^2)\\ &\qquad - (\hat{\mathbf{n}}.\boldsymbol{\xi})\,\hat{\mathbf{B}}.\operatorname{curl}\mathbf{A}\}.\end{aligned} \tag{3·14}$$

Further, employing (2·33) we obtain

$$\begin{aligned}-\int \mathrm{d}\sigma(\hat{\mathbf{n}}.\boldsymbol{\xi})\,\hat{\mathbf{B}}.\operatorname{curl}\mathbf{A} &= \int \mathrm{d}\sigma\hat{\mathbf{n}}\times\mathbf{A}.\operatorname{curl}\mathbf{A}\\ &= \int \mathrm{d}\hat{\tau}\operatorname{div}(\mathbf{A}\times\operatorname{curl}\mathbf{A})\\ &= \int \mathrm{d}\hat{\tau}\{|\operatorname{curl}\mathbf{A}|^2 - \mathbf{A}.\operatorname{curl}(\operatorname{curl}\mathbf{A})\}.\end{aligned} \tag{3·15}$$

Thus, in virtue of (2·34) the final form of δW is

$$\delta W = \delta W_F + \delta W_S + \delta W_V, \tag{3·16}$$

where δW_F is given by (3·13),

$$\delta W_V = \frac{1}{2}\int \mathrm{d}\hat{\tau}\,|\operatorname{curl}\mathbf{A}\,|^2 \tag{3·17}$$

and

$$\delta W_S = \frac{1}{2}\int \mathrm{d}\sigma\,(\mathbf{n}.\boldsymbol{\xi})^2\,\mathbf{n}.\langle\operatorname{grad}(p+\tfrac{1}{2}|\,\mathbf{B}\,|^2)\rangle. \tag{3·18}$$

With this form for δW, (3·16), the energy principle will now be extended to displacements $\boldsymbol{\xi}$ which do not satisfy the constraint equation (2·32). It will be shown that if there exist $\boldsymbol{\xi}$ and $\mathbf{A}$ which satisfy (2·33) and (2·35), but not necessarily (2·32) and (2·34), and which make δW as given by (3·16) negative, then there is a $\tilde{\boldsymbol{\xi}}$ and and $\tilde{\mathbf{A}}$ satisfying (2·32) to (2·35) which make δW negative. Note that for the unrestricted $\boldsymbol{\xi}$ and $\mathbf{A}$, δW as given by (3·6) may differ from that given by (3·16) by the addition of terms which represent the work done at the surface against the unbalanced total pressure $\langle p + \frac{1}{2} \mid \mathbf{B} \mid^2 \rangle$. Thus the form of δW given by (3·16) must be used for the extended principle.

In order to find $\tilde{\mathbf{A}}$ observe first that the Euler equation resulting from the minimization of δW_V ((3·17) with the constraint conditions (2·33) and (2·35)) is $\text{curl}^2\, \mathbf{A} = 0$ ((2·34)). Therefore, if $\mathbf{A}$ does not satisfy this equation, $\tilde{\mathbf{A}}$ can be chosen to satisfy it and certainly decrease δW_V thereby.

To complete the proof it remains to find $\tilde{\boldsymbol{\xi}}$. This is accomplished by modifying $\boldsymbol{\xi}$ by an infinitesimal amount. Let ϵ be a parameter of smallness and $\boldsymbol{\eta}$ a finite vector in the grad p direction which falls to zero in a distance ϵ as one moves normally away from the interface into the fluid. Write $\tilde{\boldsymbol{\xi}}$ as

$$\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + \epsilon\boldsymbol{\eta}. \tag{3·19}$$

To lowest order in ϵ

$$\text{div}\,(\epsilon\boldsymbol{\eta}) = \mathbf{n}.[\mathbf{n}.\text{grad}\,(\epsilon\boldsymbol{\eta})] - [\mathbf{n} \times (\mathbf{n} \times \text{grad})].(\epsilon\boldsymbol{\eta}) \sim |\,\boldsymbol{\eta}\,|, \tag{3·20}$$

since $\boldsymbol{\eta}$ changes rapidly in the normal direction. Thus $\boldsymbol{\eta}$ can be chosen so that $\tilde{\boldsymbol{\xi}}$ satisfies (2·32). Furthermore

$$\begin{aligned} \delta W\{\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\} &= \delta W\{\boldsymbol{\xi} + \epsilon\boldsymbol{\eta}, \boldsymbol{\xi} + \epsilon\boldsymbol{\eta}\} \\ &= \delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\} + O(\epsilon), \end{aligned} \tag{3·21}$$

since the integrands of $\delta W\{\boldsymbol{\xi}, \epsilon\boldsymbol{\eta}\}$ and $\delta W\{\epsilon\boldsymbol{\eta}, \epsilon\boldsymbol{\eta}\}$ are bounded and are different from zero only in a shell of thickness ϵ. Therefore, if $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ is negative ϵ can be chosen so small that $\delta W\{\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}\}$ is negative. It is clear that any $\boldsymbol{\xi}$ and $\mathbf{A}$ which do satisfy the conditions (2·32) and (2·34) can be considered to be members of the unrestricted class of $\boldsymbol{\xi}$ and $\mathbf{A}$. Thus, a necessary and sufficient condition for instability is that one can find a $\boldsymbol{\xi}$ and $\mathbf{A}$ which satisfy only (2·33) at a fluid-vacuum interface and (2·35) or (2·36) at a rigid, perfectly conducting boundary and make the potential energy, (3·16), negative. This completes the proof of the extended energy principle.

The above considerations are closely connected with Rayleigh's principle (Rayleigh 1877). In fact, it can be shown that the Euler equation of the variational principle

$$\omega^2 = \frac{\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}}{K\{\boldsymbol{\xi}, \boldsymbol{\xi}\}}, \quad \Delta\omega^2 = 0 \tag{3·22}$$

is just the eigenvalue equation (3·8). (Note that δ represents a variation due to a $\boldsymbol{\xi}$ deformation, while Δ is used to represent other variations.) If the form of δW is given by (3·6) then the variation in $\boldsymbol{\xi}$, $\Delta\boldsymbol{\xi}$, must satisfy (2·32). If, however, (3·16) is used for δW, then the variations $\Delta\mathbf{A}$ and $\Delta\boldsymbol{\xi}$ are subject only to equation (2·33), and (2·32) follows as a natural boundary condition.

The utility of Rayleigh's principle lies in the fact that when the ratio (3·22) possesses a minimum, it can be used to estimate oscillation frequencies or rates of growth of instability. For example, those displacements which make δW negative can be used as trial functions in the variational principle (3·22). Even when ω^2 is not bounded from below as is the case in certain hydromagnetic instabilities

(Kruskal & Schwarzschild 1954) Rayleigh's principle can still be employed to yield information on the structure and time constant of the eigenmodes.

In practice, the examination of the sign of δW in the energy principle is carried out in many cases by choosing a positive definite normalization condition on $\boldsymbol{\xi}$ and minimizing δW. This is formally similar to (3·22). The great advantage of the energy principle over both the normal mode technique and its equivalent, Rayleigh's principle, lies in the fact that one is not restricted to the normalization condition $K\{\boldsymbol{\xi}, \boldsymbol{\xi}\} = 1$, but can choose any convenient condition. Of course, in changing the normalization condition one loses knowledge of the exact eigenfrequencies but often gains the advantages of great analytical simplification. In § 6, there appear examples of alternative normalization conditions.

(a) *Extension of the energy principle to more general cases*

The description of a plasma given above may be inadequate if any of the conditions of validity (i) to (vii) of § 2 do not hold. In many cases of interest condition (iv), that the stress tensor be isotropic, and part of condition (vii), that the ion pressure gradient term and the gravitational potential term in Ohm's law be negligible, are not satisfied. In this section the formalism is generalized to include situations in which these conditions are no longer valid.

The new governing equations will be stated and δW derived. It will be found that the inclusion of the new terms in Ohm's law does not lead to a change in the formula for δW, when the stress tensor is isotropic.

In these more general cases, the equation of motion (2·1) is

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} - \operatorname{div} \overleftrightarrow{\mathbf{p}} - \rho \operatorname{grad} \phi \tag{3·23}$$

and the valid Ohm's law, replacing (2·3) is

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{M}{e\rho} \operatorname{div} \overleftrightarrow{\mathbf{p}}_i - \frac{M}{e} \operatorname{grad} \phi = 0, \tag{3·24}$$

where $\overleftrightarrow{\mathbf{p}}$ is the total material stress tensor and $\overleftrightarrow{\mathbf{p}}_i$ is the ion partial stress tensor.

To derive an equation of state for the case of an anisotropic stress tensor, consider situations where the magnetic field is so strong that its change over an ion Larmor radius is small. Then the matter stress tensor $\overleftrightarrow{\mathbf{p}}$ is approximately diagonal in a local Cartesian co-ordinate system one of whose axes is directed along $\mathbf{B}$, and is invariant under rotations about $\mathbf{B}$. That is, if $\mathbf{e}$ denotes a unit vector parallel to $\mathbf{B}$ and $\overleftrightarrow{\mathbf{1}}$ the unit dyadic,

$$\overleftrightarrow{\mathbf{p}} = p_\perp(\overleftrightarrow{\mathbf{1}} - \mathbf{ee}) + p_\parallel \mathbf{ee}. \tag{3·25}$$

The internal energy per unit volume is given by one-half the trace of the stress tensor. Thus, the internal energy per unit mass can be written

$$u = u_\parallel + u_\perp, \tag{3·26}$$

where

$$u_\parallel = \frac{p_\parallel}{2\rho}, \tag{3·27}$$

and

$$u_\perp = \frac{p_\perp}{2\rho}. \tag{3·28}$$

If collisions are infrequent, $u_{\parallel}$ and $u_{\perp}$ are independent. Assume that there is no flow of heat and consider an element of mass contained in the element of volume $d\tau = dL\,dS$, where dL is an element of length along $\mathbf{B}$ and dS an element of area perpendicular to $\mathbf{B}$. In a displacement $\boldsymbol{\xi}$ the associated fractional change in length along a line of force is easily seen to be

$$\frac{\delta dL}{dL} = \frac{1}{dL}\mathbf{e}.[\boldsymbol{\xi}(\mathbf{r}+\mathbf{e}dL)-\boldsymbol{\xi}(\mathbf{r})] = (\mathbf{e}.\mathrm{grad}\,\boldsymbol{\xi}).\mathbf{e}. \tag{3·29}$$

The corresponding fractional change in area perpendicular to $\mathbf{B}$ is readily computed by observing that it follows from $d\tau = dL\,dS$ and $\delta d\tau/d\tau = \mathrm{div}\,\boldsymbol{\xi}$ that

$$\frac{\delta dL}{dL}+\frac{\delta dS}{dS} = \frac{\delta d\tau}{d\tau} = \mathrm{div}\,\boldsymbol{\xi}; \tag{3·30}$$

whence

$$\frac{\delta dS}{dS} = \mathrm{div}\,\boldsymbol{\xi}-(\mathbf{e}.\mathrm{grad}\,\boldsymbol{\xi}).\mathbf{e}. \tag{3·31}$$

Thus if there is no heat flow in the course of the displacement, that is, if the displacement is locally adiabatic, one can write

$$\delta(u_{\parallel}\rho d\tau) = \delta(\tfrac{1}{2}p_{\parallel}d\tau) = -p_{\parallel}dS\,\delta dL, \tag{3·32}$$

$$\delta(u_{\perp}\rho d\tau) = \delta(p_{\perp}d\tau) = -p_{\perp}dL\,\delta dS. \tag{3·33}$$

The terms on the right above represent the external work done. From these expressions follow immediately the equations of state.

$$\left.\begin{aligned}\frac{\delta p_{\parallel}}{p_{\parallel}} &= -\mathrm{div}\,\boldsymbol{\xi}-2(\mathbf{e}.\mathrm{grad}\,\boldsymbol{\xi}).\mathbf{e},\\ \frac{\delta p_{\perp}}{p_{\perp}} &= -2\,\mathrm{div}\,\boldsymbol{\xi}+(\mathbf{e}.\mathrm{grad}\,\boldsymbol{\xi}).\mathbf{e}.\end{aligned}\right\} \tag{3·34}$$

These equations agree with those found by Chew, Goldberger & Low (1956) by an analysis of the Boltzmann equation, employing somewhat different assumptions.

In order to derive the expression for the change in $\mathbf{B}$ due to a displacement $\boldsymbol{\xi}$, consider motions about a configuration of static equilibrium. For clarity the subscript zero is reintroduced to indicate equilibrium quantities. The equilibrium electric field is

$$\mathbf{E}_0 = \frac{M}{e}\mathrm{grad}_0\,\phi_0+\frac{M}{e\rho_0}\mathrm{div}_0\,\overleftrightarrow{\mathbf{p}}_{i,0}. \tag{3·35}$$

Since $\mathbf{E}_0$ is an electrostatic field its curl must vanish which implies that the right-hand side of (3·35) is the gradient of a scalar.

Assume that (3·34) holds with $p_{\parallel}$ and $p_{\perp}$ replaced by $p_{i\parallel}$ and $p_{i\perp}$, and note that in order of magnitude $p_{i\parallel} \sim p_{i\perp} \sim \rho kT_i/M$. Then the change in magnitude of

$$(M\,\mathrm{div}\,\overleftrightarrow{\mathbf{p}}_i)/e\rho$$

in a displacement $\boldsymbol{\xi}$ from equilibrium, which is not necessarily small, is approximately

$$\frac{M}{e}\frac{kT_i}{ML}\frac{\xi}{L}, \tag{3·36}$$

where L is a characteristic length over which the various physical quantities change. The corresponding change in the magnitude of $\mathbf{v}\times\mathbf{B}$ is

$$\omega\xi B, \tag{3·37}$$

where $1/\omega$ is a characteristic time of the motion. The ratio of formula (3·36) to formula (3·37) is

$$\frac{kT_i/M}{\omega\omega_{ci}L^2}, \tag{3·38}$$

where $\omega_{ci} = eB/M$ is the ion cyclotron frequency. For many systems of interest $\omega^2L^2 \sim kT_i/M$, while $\omega \ll \omega_{ci}$ by condition (vii) of § 2. Thus the ratio (3·38) is much less than unity and the ion stress tensor has negligible effect in determining the change in **E** from its equilibrium value, although it may play an important role in determining $\mathbf{E}_0$.

The change in **B**, however, is determined from $\operatorname{curl}\dot{\mathbf{E}} = -\partial\mathbf{B}/\partial t$. Thus it follows from the Ohm's law equation (3·24), neglecting the contribution of the term in $\operatorname{div}\overleftrightarrow{\mathbf{p}}_i$ on the basis of the preceding considerations, that

$$\frac{\partial\mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}). \tag{3·39}$$

Equation (3·39), however, is precisely what one obtains on combining the induction equation (2·5) with the Ohm's law of the preceding work, (2·3). Thus, in those cases where the stress tensor is isotropic, the linearized equations governing the motion are unchanged by the inclusion in the Ohm's law of the two additional terms. Therefore, $\mathbf{F}\{\boldsymbol{\xi}\}$ and δW are also unchanged and the energy principle holds in the form previously derived.

If the stress tensor is given by (3·25) and (3·34) there exists an energy integral

$$U = \int d\tau\{\tfrac{1}{2}\rho\,|\,\mathbf{v}\,|^2 + \tfrac{1}{2}\,|\,\mathbf{B}\,|^2 + p_\perp + \tfrac{1}{2}p_\parallel + \rho\phi\}, \tag{3·40}$$

while (3·34) (3·39), and the law of conservation of mass $\dot{\rho} = -\rho\operatorname{div}\mathbf{v}$ permit one to express $\overleftrightarrow{\mathbf{p}}$, **B** and ρ in terms of their initial values and $\boldsymbol{\xi}$. The expressions do not involve $\dot{\boldsymbol{\xi}}$. Thus, since the system is conservative there must exist a potential energy δW quadratic in $\boldsymbol{\xi}$ which implies as before that the associated first order force $\mathbf{F}\{\boldsymbol{\xi}\}$ is self-adjoint. The energy principle is, therefore, still valid and stability can be determined by examining the sign of the new δW which is given by

$$\begin{aligned}
\delta W &= -\frac{1}{2}\int d\tau\,\boldsymbol{\xi}\,.\,\mathbf{F}\{\boldsymbol{\xi}\} \\
&= \frac{1}{2}\int d\hat{\tau}\,|\operatorname{curl}\mathbf{A}\,|^2 \\
&\quad -\frac{1}{2}\int d\sigma\{(\mathbf{n}\,.\,\boldsymbol{\xi})^2\,\mathbf{n}\,.\,\langle\operatorname{grad}(p_\perp + \tfrac{1}{2}\,|\,\mathbf{B}\,|^2)\rangle \\
&\qquad\qquad + \mathbf{e}\,.\,\boldsymbol{\xi}(p_\parallel - p_\perp)\,\mathbf{n}\,.\,[(\mathbf{e}\,.\operatorname{grad})\,\boldsymbol{\xi} - (\boldsymbol{\xi}\,.\operatorname{grad})\,\mathbf{e}]\} \\
&\quad +\frac{1}{2}\int d\tau\{|\,\mathbf{Q}\,|^2 - \mathbf{j}\,.\,\mathbf{Q}\times\boldsymbol{\xi} + \tfrac{5}{3}p_\perp(\operatorname{div}\boldsymbol{\xi})^2 + (\operatorname{div}\boldsymbol{\xi})\,(\boldsymbol{\xi}\,.\operatorname{grad})\,p_\perp \\
&\quad + \tfrac{1}{3}p_\perp[\operatorname{div}\boldsymbol{\xi} - 3q]^2 + q\operatorname{div}[\boldsymbol{\xi}(p_\parallel - p_\perp)] - [\operatorname{div}(\rho\boldsymbol{\xi})]\,\boldsymbol{\xi}\,.\operatorname{grad}\phi \\
&\quad - (p_\parallel - p_\perp)\,[\mathbf{e}\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e} - \boldsymbol{\xi}\,.\,(\operatorname{grad}\mathbf{e})\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e} \\
&\quad - 4q^2 + \mathbf{e}\,.\,(\operatorname{grad}\boldsymbol{\xi})\,.\,(\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi}) - \boldsymbol{\xi}\,.\,(\operatorname{grad}\mathbf{e})\,.\,(\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi})]\},
\end{aligned} \tag{3·41}$$

where $q = (\mathbf{e}\,.\operatorname{grad}\boldsymbol{\xi})\,.\,\mathbf{e}$ and the subscript zero distinguishing equilibrium quantities has been suppressed.

The boundary condition on $\mathbf{A}$ remains as before, (2·33). The jump condition on the pressure, (2·8), is replaced by

$$\langle p_\perp + \tfrac{1}{2} |\mathbf{B}|^2 \rangle = 0. \tag{3·42}$$

In some cases collisions are sufficiently frequent to yield an isotropic stress tensor for the equilibrium, but the collision time is much greater than an oscillation or instability time. Under such circumstances the stress tensor will not remain isotropic in the course of a motion but will be determined by (3·34), with $p_\parallel = p_\perp = p$. Expression (3·41) for δW then differs by a positive definite term from the corresponding equation (3·13) for the case where the stress tensor remains isotropic in the course of a motion with $\gamma = \frac{5}{3}$. Hence, the equilibrium is at least as stable.

(b) *Comparison theorems*

There are various comparison theorems which follow from the energy principle. Two examples will now be given.

Consider a system (I), a part of which is a vacuum region (a). Compare this with a system (II), which in the equilibrium state is identical with (I), except that the part corresponding to (a) is a zero-pressure plasma. Then if system (II) is unstable, so is system (I). To demonstrate this it is merely necessary to note that the expressions for δW for the two systems differ only in that the vacuum contribution $\frac{1}{2}\int d\tau\, |\operatorname{curl} \mathbf{A}|^2$ for region (a) of system (I) is replaced by $\frac{1}{2}\int |\operatorname{curl}(\boldsymbol{\xi} \times \mathbf{B})|^2 d\tau$ for system (II). Suppose $\boldsymbol{\xi}_{II}$ and $\mathbf{A}_{II}$ are trial functions which make the change in potential energy for system (II) negative. Then for system (I) choose $\mathbf{A}_I = \mathbf{A}_{II}$ and $\boldsymbol{\xi}_I = \boldsymbol{\xi}_{II}$ except in region (a) and there choose $\mathbf{A}_I = \boldsymbol{\xi}_{II} \times \mathbf{B}$, which is a valid trial function, since it satisfies the boundary conditions on $\mathbf{A}$. This choice makes δW for (I) also negative.

A second comparison theorem is established by considering two equilibria; case (I), a fluid region in contact with a surrounding vacuum region which in turn is enclosed by a rigid perfectly conducting wall; case (II), a fluid region which is identical with the fluid region of I, but is in contact with a surrounding vacuum region enclosed in a rigid perfectly conducting wall which either coincides with or is exterior to that of (I). Assume further that all equilibrium quantities are identical in the common regions of (I) and (II).

Suppose that vector fields $\boldsymbol{\xi}$ and $\mathbf{A}$ have been found which make δW negative for case (I). The vector potential $\mathbf{A}$ can be assumed to vanish identically on the rigid perfectly conducting wall enclosing (I) because of (2·35) and the fact that an arbitrary gradient can be added to $\mathbf{A}$ without changing δW. Clearly the same vector fields can be employed as trial functions for (II) with A chosen to be zero in any regions not common to both systems. Thus system (II) is certainly no more stable than (I).

4. Application of the energy principle

(a) *Procedure*

The energy principle shows that the question of stability of an equilibrium situation is reduced to an examination of the sign of $\delta W\{\boldsymbol{\xi}, \boldsymbol{\xi}\}$ for arbitrary displacements $\boldsymbol{\xi}$. For some equilibria physical reasoning leads to $\boldsymbol{\xi}$'s which make

$\delta W\{\xi, \xi\}$ negative, and thus settles the question of stability in a simple manner. An example of this kind is given in § 5. In general, however, it is not possible immediately to exhibit such a ξ. In this case a procedure is needed for examining $\delta W\{\xi, \xi\}$ for all admissible ξ's in a systematic fashion. One tries to make $\delta W\{\xi, \xi\}$ as small as possible. Since it is a homogeneous quadratic form in ξ, one must introduce a condition to keep its values bounded from below. This condition can be chosen in any convenient way so long as it does not affect the sign of $\delta W\{\xi, \xi\}$. In particular it can be chosen to lead to analytical simplicity in the minimization. For example, one can impose normalization requirements like $\int d\tau_0 \rho_0 \xi^2 = 1$, or alternatively one can prescribe $\mathbf{n}_0 . \xi$ on the fluid vacuum boundary (where a subscript zero as usual denotes equilibrium quantities). In the latter case, it is, of course, necessary to minimize separately for all admissible prescriptions of $\mathbf{n}_0 . \xi$.

Consider a plasma surrounded by a vacuum region. A convenient program for minimization consists of first examining ξ's which do not move the interface (i.e. $\mathbf{n}_0 . \xi = 0$ on the interface). Note that with this boundary condition the surface terms do not contribute to δW and the non-negative vacuum term is minimized to zero by choosing $\mathbf{A} = 0$. If δW can be made negative, be it by inspection or by choosing a normalization condition and minimizing, then the equilibrium is unstable.

Suppose, however, δW is non-negative with the above boundary condition $\mathbf{n}_0 . \xi = 0$. The equilibrium still may not be stable since displacements which move the boundary may yield a decrease in potential energy. In this case it is convenient to proceed by prescribing $\mathbf{n}_0 . \xi$ (not everywhere zero) on the fluid-vacuum boundary, and minimizing δW_V and δW_F separately. No volume condition like $\int d\tau_0 \rho_0 \xi^2 = 1$ is imposed here. Since δW_V is a non-negative form whose Euler equation is

$$\text{curl}_0 \, \text{curl}_0 \, \mathbf{A} = 0 \tag{4·1}$$

it obviously has a minimum.

Assume further, as is often true in practice, that there is a displacement ξ which makes δW_F stationary subject to a given prescription of $\mathbf{n}_0 . \xi$. Then this stationary value must be an absolute minimum and thus unique. To show this let η be any displacement which satisfies the boundary condition $\mathbf{n}_0 . \eta = 0$. Then

$$\delta W_F\{\xi+\eta, \xi+\eta\} = \delta W_F\{\xi, \xi\} + 2\delta W_F\{\xi, \eta\} + \delta W_F\{\eta, \eta\}. \tag{4·2}$$

The assumption that ξ makes δW_F stationary requires that $\delta W_F\{\xi, \eta\} = 0$, and leads to the Euler equation

$$\mathbf{F}\{\xi\} = 0. \tag{4·3}$$

Now, since $\mathbf{n}_0 . \eta = 0$, $\delta W_F\{\eta, \eta\}$ is non-negative by supposition. Thus $\delta W\{\xi, \xi\}$ is a minimum.

Form the scalar product of (4·1) with $\mathbf{A}$, and of (3·3) with ξ, and integrate over their respective volumes. The resulting minimum potential energy, subject to the prescribed boundary values $\mathbf{n}_0 . \xi$, is

$$\delta W = \frac{1}{2}\int d\sigma_0 \mathbf{n}_0 . \xi\{\gamma p_0 \,\text{div}_0\, \xi - \mathbf{B}_0 . \mathbf{Q}_0 - \mathbf{B}_0 . (\xi . \text{grad}_0 \mathbf{B}_0) + \hat{\mathbf{B}}_0 . \text{curl}\, \mathbf{A} + \hat{\mathbf{B}}_0 . (\xi . \text{grad}_0 \hat{\mathbf{B}}_0)\}. \tag{4·4}$$

This expression, of course, represents the work done against the unbalanced first order total pressure $\langle p + \frac{1}{2} \mid \mathbf{B} \mid^2 \rangle$ in a displacement of the boundary. Note that δW in (4·4) is a functional of $\mathbf{n}_0 . \boldsymbol{\xi}$. The program is completed by minimizing (4·4) with respect to $\boldsymbol{\xi} . \mathbf{n}_0$.

(b) *A physical interpretation*

The problem of minimizing the volume contribution δW_F subject to the boundary condition $\mathbf{n}_0 . \boldsymbol{\xi} = 0$, under a particular normalization, yields conditions of physical interest on the minimizing $\boldsymbol{\xi}$ when $\nabla \phi \equiv 0$. These conditions are that to first order in $\boldsymbol{\xi}$ the fields $\mathbf{j}$ and $\mathbf{B}$ are tangent to the surfaces p = constant. That this is true to zero order in $\boldsymbol{\xi}$, that is, for the equilibrium quantities, follows from (2·20).

The choice of normalization for the demonstration is motivated by the fact that it is possible by judicious integration by parts to write

$$\delta W_F = \frac{1}{2} \int \mathrm{d}\tau_0 \{ \mid \mathbf{Q}_0 + \mathbf{n}_0 . \boldsymbol{\xi} \, \mathbf{j}_0 \times \mathbf{n}_0 \mid^2 + \gamma p_0 (\mathrm{div}_0 \, \boldsymbol{\xi})^2 - 2(\mathbf{n}_0 . \boldsymbol{\xi})^2 \mathbf{j}_0 \times \mathbf{n}_0 . (\mathbf{B}_0 . \mathrm{grad}_0 \, \mathbf{n}_0) \}, \tag{4·5}$$

where $\mathbf{n}_0$ is the unit vector normal to the surface p_0 = constant. It is obvious from (4·5) that a normalizing condition involving $\mathbf{n}_0 . \boldsymbol{\xi}$ alone (e.g. $\int \mathrm{d}\tau_0 \, \rho_0 (\mathbf{n}_0 . \boldsymbol{\xi})^2 = 1$) should be sufficient to bound $\delta W_F \{ \boldsymbol{\xi}, \boldsymbol{\xi} \}$ from below. Let $\boldsymbol{\xi}$ minimize δW with such a normalizing condition. Any small change in $\boldsymbol{\xi}$, $\Delta \boldsymbol{\xi}$, must leave δW stationary, if it leaves the norm stationary. From the self-adjoint nature of $\mathbf{F}$, it follows that

$$\Delta [\delta W] = - \int \mathrm{d}\tau_0 \Delta \boldsymbol{\xi} . \mathbf{F} \{ \boldsymbol{\xi} \} = 0. \tag{4·6}$$

Consider $\Delta \boldsymbol{\xi}$'s of the form

$$\Delta \boldsymbol{\xi} = \Delta b \mathbf{j}_0 + \Delta c \mathbf{B}_0, \tag{4·7}$$

where Δb and Δc are arbitrary since the normalization condition involves only $\boldsymbol{\xi} . \mathrm{grad}_0 \, p_0$, and $\mathbf{j}_0$ and $\mathbf{B}_0$ are orthogonal to $\mathrm{grad}_0 \, p_0$. Then it follows that the coefficients of Δb and Δc in the integrand of equation (4·6) must separately vanish. Now $\mathbf{F}\{\boldsymbol{\xi}\} = \{ - \mathrm{grad} \, p + \mathbf{j} \times \mathbf{B} \}_1$, where the subscript unity means the part first order in $\boldsymbol{\xi}$, so

$$\mathbf{B}_0 . \{ - \mathrm{grad} \, p + \mathbf{j} \times \mathbf{B} \}_1 = 0, \tag{4·8}$$

$$\mathbf{j}_0 . \{ - \mathrm{grad} \, p + \mathbf{j} \times \mathbf{B} \}_1 = 0. \tag{4·9}$$

Note, however, that it follows on taking the first order part of the identities $\mathbf{B} . \mathbf{j} \times \mathbf{B} = 0$ and $\mathbf{j} . \mathbf{j} \times \mathbf{B} = 0$, and using $\mathbf{j}_0 \times \mathbf{B}_0 = (\mathrm{grad} \, p)_0$, that

$$\mathbf{B}_0 . (\mathbf{j} \times \mathbf{B})_1 + \mathbf{B}_1 . (\mathrm{grad} \, p)_0 = 0, \tag{4·10}$$

$$\mathbf{j}_0 . (\mathbf{j} \times \mathbf{B})_1 + \mathbf{j}_1 . (\mathrm{grad} \, p)_0 = 0. \tag{4·11}$$

Thus if one subtracts (4·10) and (4·11), respectively, from (4·8) and (4·9), there results, correct to first order in $\boldsymbol{\xi}$,

$$(\mathbf{B}_0 + \mathbf{B}_1) . [(\mathrm{grad} \, p)_0 + (\mathrm{grad} \, p)_1] = 0, \tag{4·12}$$

$$(\mathbf{j}_0 + \mathbf{j}_1) . [(\mathrm{grad} \, p)_0 + (\mathrm{grad} \, p)_1] = 0. \tag{4·13}$$

Equations (4·12) and (4·13) express the conditions stated earlier, that to first order in $\boldsymbol{\xi}$ the fields $\mathbf{j}$ and $\mathbf{B}$ are tangent to the surfaces p = constant.

After some manipulation, (4·12) (or equivalently (4·8)) can be rewritten in the form

$$\mathbf{B}_0 \cdot \text{grad}_0 \, \text{div}_0 \, \boldsymbol{\xi} = 0, \tag{4·14}$$

which is often useful in practice.

5. Stability of a plasma with no internal magnetic field

Consider a plasma in which the magnetic field vanishes and the pressure is constant and outside which there is a vacuum region with a magnetic field. Let $\phi = 0$. It was suggested by E. Teller (1954, private communication) on intuitive grounds that if the lines of force on the interface are anywhere concave to the plasma the state is unstable to local displacements. This is readily demonstrated using the energy principle.

Choose a divergence-free displacement $\boldsymbol{\xi}$ so that

$$2\delta W = \int d\hat{\tau} \, | \, \text{curl} \, \mathbf{A} \, |^2 - \frac{1}{2} \int d\sigma \, (\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \, \hat{\mathbf{n}} . \text{grad} \, | \, \hat{\mathbf{B}} \, |^2, \tag{5·1}$$

where $\hat{\mathbf{n}}$ is the normal to the interface pointing towards the plasma. Denote by $\mathbf{R}$ the vector from a point on a line of force to the centre of curvature of the line. Since, with $|\mathbf{R}| = R$,

$$\tfrac{1}{2} \hat{\mathbf{n}} . \text{grad} \, | \, \mathbf{B} \, |^2 = \hat{\mathbf{n}} . \mathbf{R} \frac{|\mathbf{B}|^2}{R^2}, \tag{5·2}$$

the surface term in (5·1) is negative or positive according to whether or not $\mathbf{R}$ points towards the plasma. If $\mathbf{R}$ everywhere points away from the plasma, δW is obviously positive for all $\boldsymbol{\xi}$ and $\mathbf{A}$ (even if div $\boldsymbol{\xi} \neq 0$) and the system is stable.

Consider a point on the interface where $\mathbf{R}$ is directed towards the plasma and construct a local Cartesian co-ordinate system in a small region about this point, with the z axis normal to the surface and pointing into the vacuum, and the x axis in the direction of $\mathbf{B}$. Choose the trial displacement $\boldsymbol{\xi}$ so that

$$\xi_z(x, y, 0) = \xi_0 f(x, y) \sin ky, \tag{5·3}$$

where f is a function of order unity which falls to zero in the small distance $a \ll R$ and where $ka^2 \gg R$. Choose also the trial vector potential

$$\mathbf{A}(x, y, z) = f(x, y) \, \text{grad} \left(\frac{\xi_0 B}{k} \cos ky \, e^{-kz} \right), \tag{5·4}$$

which satisfies boundary condition (2·33) where

$$\mathbf{B} = B \mathbf{e}_x. \tag{5·5}$$

These choices make the vacuum contribution to δW negligible compared to the surface term. For

$$\int d\hat{\tau} \, | \, \text{curl} \, \mathbf{A} \, |^2 = \int d\hat{\tau} \left\{ \text{grad} f \times \text{grad} \left[\frac{\xi_0 B}{k} \cos ky \, e^{-kz} \right] \right\}^2$$

$$\approx \int d\hat{\tau} \, | \, \text{grad} f \, |^2 \, \xi_0^2 B^2 \, e^{-2kz} \approx \frac{\xi_0^2 B^2}{2k}, \tag{5·6}$$

while

$$\int d\sigma \, (\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \, \hat{\mathbf{n}} . \mathbf{R} \frac{|\hat{\mathbf{B}}|^2}{R^2} \approx \frac{1}{R} \xi_0^2 B^2 a^2. \tag{5·7}$$

Therefore, δW is negative and by the energy principle the system is unstable.

Note that the deformation which produces instability tends to flute the surface along the lines of force. This moves some of the magnetic lines of force into a region previously occupied by matter and thus shortens them while only slightly bending them. The result is a decrease in the magnetic energy with no change in the gas energy.

Similar results have been obtained independently by H. Grad (1955) and C. Longmire (1955) (both private communications).

To estimate the rate of growth of this instability in the plasma choose the displacement

$$\xi_x = 0, \quad \xi_y = \xi_0 \surd f \cos ky\, e^{kz}, \quad \xi_z = \xi_0 \surd f \sin ky\, e^{kz}. \tag{5·8}$$

This $\boldsymbol{\xi}$ satisfies $\operatorname{div} \boldsymbol{\xi} = 0$ to order $(ka)^{-1}$. Then the kinetic energy form is

$$2K = \int d\tau \rho \,|\, \boldsymbol{\xi} \,|^2 \approx \frac{\rho \xi_0^2 a^2}{2k}$$

and

$$\omega^2 = \frac{\delta W}{K} \approx -\frac{2B^2}{\rho} \frac{k}{R}. \tag{5·9}$$

This is unbounded as k approaches infinity.

Gravitational effects are readily included in this case if we assume the fluid to have constant density but varying pressure in the equilibrium state. This situation is an extension of the hydromagnetic Rayleigh–Taylor problem (Kruskal & Schwarzschild 1954) to an arbitrary interface. Choose the trial functions $\boldsymbol{\xi}$ and $\mathbf{A}$ as before. The surface term in (5·1) is modified by the addition of the term

$$\int d\sigma \,(\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \,\hat{\mathbf{n}} . \operatorname{grad} p \tag{5·10}$$

which, in virtue of the equilibrium relation

$$\operatorname{grad} p = -\rho \operatorname{grad} \phi, \tag{5·11}$$

becomes

$$-\int d\sigma \,(\hat{\mathbf{n}} . \boldsymbol{\xi})^2 \rho \hat{\mathbf{n}} . \operatorname{grad} \phi. \tag{5·12}$$

The calculation now goes through as before and the situation is unstable if

$$\hat{\mathbf{n}} . \left[\mathbf{R} \frac{|\,\hat{\mathbf{B}}\,|^2}{R^2} + \rho \operatorname{grad} \phi\right] > 0 \tag{5·13}$$

anywhere on the boundary. In the case of a plane interface $\mathbf{R}$ is infinite and the familiar hydromagnetic Rayleigh–Taylor instability criterion is recovered.

6. Stability of an axisymmetric system

A more general case than that treated in the previous section occurs when a magnetic field may be present in the plasma. This situation can be treated exactly for two simple types of axisymmetric equilibrium situations. It is assumed that gravitational effects are negligible ($\phi = 0$).

The first consists of a longitudinal current giving rise to a toroidal magnetic field whose pressure supports a radial material pressure gradient. This is the well-known pinch effect (see, for example, Kruskal & Schwarzschild 1954).

The second consists of longitudinal and radial magnetic fields produced by currents in the azimuthal direction. Again, a radial material pressure gradient is supported by the magnetic field. The plasma is assumed to be in contact with a rigid perfectly conducting wall. This equilibrium is studied here. It is shown that it is possible to reduce the problem of stability to the consideration of an ordinary second-order differential equation of the Sturm–Liouville type. In fact, virtually all that is necessary is to find the number of negative eigenvalues which this equation possesses. In certain limiting cases one can further express the criterion for stability in terms of simple properties of the equilibrium.

Note that in the previous problem of § 5 either δW is obviously positive definite or one can easily display trial functions $\boldsymbol{\xi}$ and $\mathbf{A}$ which make it negative. In the problem of this section, however, it is necessary to examine the sign of δW for all possible displacements $\boldsymbol{\xi}$. This is accomplished by first writing δW in a co-ordinate system natural to the problem and then successively minimizing with respect to the components of the vector $\boldsymbol{\xi}$.

The equilibrium vector potential $\mathbf{A}_0$ in a fluid of this type (which is to be distinguished from the first order vacuum vector potential $\mathbf{A}$ previously introduced) has only an azimuthal component, since the current density j is itself azimuthal. Therefore, if in cylindrical co-ordinates (r, θ, z) one writes $\psi = rA_0(r, z)$, then

$$\mathbf{B} = \operatorname{curl}(\mathbf{e}_\theta \psi/r) = -(1/r)\,\mathbf{e}_\theta \times \operatorname{grad}\psi. \tag{6·1}$$

It follows from equation (6·1) that $\mathbf{B}.\operatorname{grad}\psi = 0$. Thus the lines of force lie in the surfaces ψ = constant and also in the planes θ = constant. Moreover, if one chooses $\psi(0, z) = 0$, it is readily demonstrated that the magnetic flux interior to the surface ψ = constant is $2\pi\psi$.

Because of this flux property, it is convenient to employ ψ as a co-ordinate. In order to retain an orthogonal co-ordinate system, introduce a function χ whose level surfaces are perpendicular to the surfaces ψ = constant and θ = constant. Choose χ so that the set (ψ, θ, χ) forms a right-handed orthogonal system. Note that the volume element in this co-ordinate system is

$$d\tau = J\,d\psi\,d\theta\,d\chi, \tag{6·2}$$

where

$$1/J = B\,|\operatorname{grad}\chi| = \operatorname{grad}\psi.\operatorname{grad}\theta \times \operatorname{grad}\chi. \tag{6·3}$$

Thus

$$\operatorname{grad} = rB\mathbf{e}_\psi \frac{\partial}{\partial\psi} + \frac{1}{r}\,\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \frac{1}{JB}\,\mathbf{e}_\chi \frac{\partial}{\partial\chi}, \tag{6·4}$$

where

$$\mathbf{e}_\psi = \frac{\operatorname{grad}\psi}{|\operatorname{grad}\psi|}, \tag{6·5}$$

$$\mathbf{e}_\theta = \frac{\operatorname{grad}\theta}{|\operatorname{grad}\theta|}, \tag{6·6}$$

$$\mathbf{e}_\chi = \frac{\operatorname{grad}\chi}{|\operatorname{grad}\chi|}. \tag{6·7}$$

Then, by (2·21) and (2·20),

$$\mathbf{j} = -\mathbf{e}_\theta \frac{r}{J} \frac{\partial}{\partial\psi}(JB^2) = j\mathbf{e}_\theta \tag{6·8}$$

and

$$\operatorname{grad} p = \mathbf{j} \times \mathbf{B} = -\mathbf{e}_\psi \frac{rB}{J} \frac{\partial}{\partial\psi}(JB^2). \tag{6·9}$$

Thus the pressure p is a function of ψ alone and if differentiation with respect to ψ is denoted by a prime, (6·9) can be written

$$(\ln JB^2)' = -p'/B^2. \tag{6·10}$$

Therefore
$$J = \frac{1}{B^2}\exp\left\{-\int_0^{\psi} \mathrm{d}\psi \frac{p'}{B^2}\right\}, \tag{6·11}$$

where the constant of integration (which is an arbitrary function of χ) has incidentally been chosen to make χ reduce to the magnetic scalar potential when $p = 0$.

Note that it follows from (6·8) and (6·9) that

$$p' = j/r \tag{6·12}$$

and j/r is constant along a line of force.

Using these results the potential energy for the system is

$$\begin{aligned}\delta W &= \delta W_F \\ &= \frac{1}{2}\int \mathrm{d}\psi \mathrm{d}\theta \mathrm{d}\chi J\left\{\left[\frac{1}{rBJ}\frac{\partial}{\partial\chi}(rB\xi_\psi)\right]^2 + \left[\frac{r}{J}\frac{\partial}{\partial\chi}\left(\frac{\xi_\theta}{r}\right)\right]^2\right. \\ &\quad + B^2\left[\frac{\partial}{\partial\psi}(rB\xi_\psi) + \frac{\partial}{\partial\theta}\left(\frac{\xi_\theta}{r}\right)\right]^2 \\ &\quad + p'rB\xi_\psi\left[\frac{\partial}{\partial\psi}(rB\xi_\psi) + \frac{\partial}{\partial\theta}\left(\frac{\xi_\theta}{r}\right)\right] \\ &\quad + \frac{\gamma p}{J^2}\left[\frac{\partial}{\partial\psi}(rB\xi_\psi J) + \frac{\partial}{\partial\theta}\left(\frac{J\xi_\theta}{r}\right) + \frac{\partial}{\partial\chi}\left(\frac{\xi_\chi}{B}\right)\right]^2 \\ &\quad + \frac{p'rB\xi_\psi}{J}\left[\frac{\partial}{\partial\psi}(rB\xi_\psi J) + \frac{\partial}{\partial\theta}\left(\frac{J\xi_\theta}{r}\right)\right] \\ &\quad \left. + \frac{1}{J}\frac{\partial}{\partial\chi}(p'\xi_\chi\xi_\psi r)\right\}.\end{aligned} \tag{6·13}$$

Assume that the equilibrium quantities appearing in (6·13) are periodic over some fundamental period in χ which is equivalent to periodicity in z and also impose the boundary condition that $\boldsymbol{\xi}$ be periodic in χ over this period. All definite integrals with respect to χ are to be understood as extended over this period. The last term in (6·13) then integrates to zero.

Now proceed to minimize δW over all displacements $\boldsymbol{\xi}$. First note that the integrand in (6·13) depends on θ only via $\boldsymbol{\xi}$. This suggests Fourier analysis of $\boldsymbol{\xi}$ with respect to θ. Write $\boldsymbol{\xi}$ in the form

$$\begin{aligned}\xi_\psi &= \sum_{m=0}^{\infty}\frac{1}{rB}X_m(\psi,\chi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\theta, \\ \xi_\theta &= \sum_{m=1}^{\infty}\frac{r}{m}Y_m(\psi,\chi)\begin{Bmatrix}\sin\\ -\cos\end{Bmatrix}m\theta + \xi_\theta^{(0)}(\psi,\chi), \\ \xi_\chi &= \sum_{m=0}^{\infty}BZ_m(\psi,\chi)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\theta.\end{aligned} \tag{6·14}$$

The potential energy δW is to be minimized over the set $(X_m, Y_m, Z_m, \xi_\theta^{(0)})$.

Upon integration with respect to θ, the cross terms of the double series vanish and

$$\delta W = \delta W_0 + 2\sum_{m=1}^{\infty} \delta W_m, \tag{6·15}$$

where

$$\begin{aligned}\delta W_m = \tfrac{1}{2}\pi\int d\chi\, d\psi\Big\{&\frac{1}{r^2B^2J}\left(\frac{\partial X_m}{\partial\chi}\right)^2 + \frac{1}{m^2}\frac{r^2}{J}\left(\frac{\partial Y_m}{\partial\chi}\right)^2 \\ &+ B^2J\left(\frac{\partial X_m}{\partial\psi}+Y_m\right)^2 + p'JX_m\left(\frac{\partial X_m}{\partial\psi}+Y_m\right) \\ &+\frac{\gamma p}{J}\left[\frac{\partial}{\partial\psi}(JX_m)+JY_m+\frac{\partial Z_m}{\partial\chi}\right]^2 \\ &+p'X_m\left(J\frac{\partial X_m}{\partial\psi}+X_m\frac{\partial J}{\partial\psi}+JY_m\right)\Big\},\end{aligned} \tag{6·16}$$

and $\frac{1}{2}\delta W_0$ is obtained by replacing Y_m in (6·16) by $m\xi_\theta^{(0)}/r$ and setting $m = 0$.

Since for each m, δW_m depends only on the set (X_m, Y_m, Z_m), it can be varied independently. It is clear from (6·16) that if δW_m can be made negative then δW_{m+1} can also be made negative. Thus it suffices to consider only the limiting case $m = \infty$. Do so and suppress the subscript ∞. After some algebraic manipulation, (6·16) becomes

$$\begin{aligned}\delta W = \tfrac{1}{2}\pi\int d\psi\, d\chi\, J\Big\{&\frac{1}{r^2B^2J^2}\left(\frac{\partial X}{\partial\chi}\right)^2 + p'X^2\frac{\partial \ln J}{\partial\psi} - \frac{p'^2X^2}{B^2} \\ &+\frac{1}{B^{-2}+(\gamma p)^{-1}}\left(X\frac{\partial \ln J}{\partial\psi}+\frac{1}{J}\frac{\partial Z}{\partial\chi}-\frac{p'X}{B^2}\right)^2 \\ &+(B^2+\gamma p)\left[Y+\frac{\partial X}{\partial\psi}+\frac{X(p'+\gamma p\,\partial\ln J/\partial\psi)+(\gamma p/J)\,\partial Z/\partial\chi}{B^2+\gamma p}\right]^2\Big\}.\end{aligned} \tag{6·17}$$

For arbitrary fixed trial functions X and Z the expression above is minimized with respect to Y by choosing

$$-Y = \frac{\partial X}{\partial\psi} + \left[X\left(p'+\gamma p\frac{\partial \ln J}{\partial\psi}\right)+\frac{\gamma p}{J}\frac{\partial Z}{\partial\chi}\right](B^2+\gamma p)^{-1}, \tag{6·18}$$

which makes

$$\delta W = \tfrac{1}{2}\pi\int d\psi\, d\chi\left\{\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial\chi}\right)^2 + p'DX^2J + \frac{J}{B^{-2}+(\gamma p)^{-1}}\left[XD+\frac{1}{J}\frac{\partial Z}{\partial\chi}\right]^2\right\}, \tag{6·19}$$

where

$$\begin{aligned}D = \frac{\partial \ln J}{\partial\psi} - \frac{p'}{B^2} &= -\frac{2}{B^2}\frac{\partial}{\partial\psi}(p+\tfrac{1}{2}B^2) \\ &= -\frac{2}{rB^3}\mathbf{e}_\psi.[\mathbf{B}.\mathrm{grad}\,\mathbf{B}]\end{aligned} \tag{6·20}$$

is positive or negative at a point χ, ψ according to whether the line of force through that point is concave or convex towards the side of smaller ψ. Consequently, the system can be unstable only if somewhere a line of force is concave toward the side of larger p.

Equation (6·18) corresponds to (4·13) of the general minimization scheme, the content of which is that the minimizing displacement is such that the perturbed current density $\mathbf{j}$ lies in the perturbed constant pressure surfaces.

Next, the Euler equation resulting from minimizing (6·19) with respect to Z for fixed X reads

$$\frac{\partial}{\partial\chi}\left[\frac{1}{B^{-2}+(\gamma p)^{-1}}\left(XD+\frac{1}{J}\frac{\partial Z}{\partial\chi}\right)\right] = 0. \tag{6·21}$$

This equation corresponds to (4·12), the content of which is that the perturbed lines of force lie in the perturbed constant pressure surfaces. Equation (6·21) yields on integration with respect to χ

$$\frac{\partial Z}{\partial\chi} = \left(\frac{1}{B^2}+\frac{1}{\gamma p}\right)Jf(\psi)-JDX. \tag{6·22}$$

The constant of integration $f(\psi)$ is determined by integrating (6·22) with respect to χ, namely,

$$f(\psi) = \frac{2\pi}{L'+V'/\gamma p}\int d\chi JDX, \tag{6·23}$$

where

$$L' = 2\pi\int d\chi\frac{J}{B^2}, \quad V' = 2\pi\int d\chi J.$$

Note that $V'd\psi$ is the volume contained between two neighbouring constant ψ surfaces.

The minimum δW now is

$$\delta W = \tfrac{1}{2}\pi\int d\psi d\chi\left\{\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial\chi}\right)^2+p'JDX^2\right\}+\frac{1}{4}\int d\psi f^2\left(L'+\frac{V'}{\gamma p}\right). \tag{6·24}$$

The integrands above do not contain any derivatives of X with respect to ψ. Thus one can consider ψ to be merely a parameter and write

$$\delta W = \int d\psi\,\delta W(\psi), \tag{6·25}$$

where $\delta W(\psi)$ depends only on the values of X on the surface ψ. Consequently δW can be made negative if and only if $\delta W(\psi)$ can be made negative for some value of ψ.

As in § 4, it is necessary to normalize X to achieve a well-posed minimum problem. An analytically simple normalizing condition is

$$H = \tfrac{1}{2}\pi\int d\chi JX^2 = 1.$$

The minimization of $\delta W(\psi)$ under this normalization is equivalent to minimizing

$$\Lambda = \frac{\delta W(\psi)}{H} = \frac{\frac{1}{2\pi}\left(L'+\frac{V'}{\gamma p}\right)f^2+\int d\chi\left[\frac{1}{r^2B^2J}\left(\frac{\partial X}{\partial\chi}\right)^2+p'JDX^2\right]}{\int d\chi JX^2}. \tag{6·26}$$

Note that L', V' and J are all positive and only the term involving D can make Λ negative. It is possible to derive a sufficient condition for instability from (6·26) by choosing X to be constant in χ. Then

$$\Lambda = \frac{\gamma p(V''-p'L')(V''/V'+p'/\gamma p)}{V'+\gamma pL'}, \tag{6·27}$$

and if for any value of ψ this expression is negative the system is unstable.

In certain limiting cases it is possible to derive necessary and sufficient stability criteria directly from (6·26). In general, however, one must proceed with the formal minimization program.

The Euler equation resulting from the minimization of Λ is

$$\frac{\partial}{\partial\chi}\left(\frac{1}{r^2B^2J}\frac{\partial X}{\partial\chi}\right)+(\Lambda-p'D)JX = JDf, \tag{6·28}$$

where the variation in f has been computed from (6·23).

It is possible to derive from (6·28) certain general criteria for stability by expanding its solutions in terms of the eigenfunctions X_j of the Sturm–Liouville equation

$$\frac{\partial}{\partial\chi}\left(\frac{1}{r^2B^2J}\frac{\partial X_j}{\partial\chi}\right)+(\lambda_j-p'D)X_jJ = 0, \tag{6·29}$$

obtained by omitting the integral on the right-hand side of (6·28).

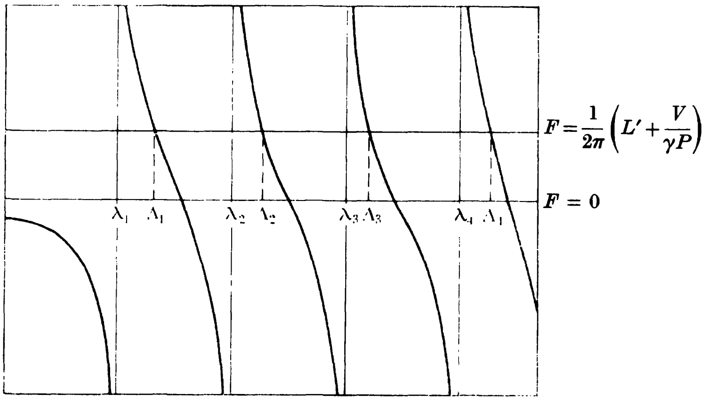

FIGURE 1. Schematic plot of $F(\Lambda)$ against Λ.

By the Sturmian theory (Ince 1944) the X_j comprise a complete set of eigen functions with associated eigenvalues λ_j. The λ_j are all distinct and can be arranged in an infinite increasing sequence $\lambda_1, \lambda_2, \ldots$. Note that the X_j can be normalized such that

$$\int d\chi X_i X_j J = \delta_{ij}. \tag{6·30}$$

Thus one can write

$$X = \Sigma b_j X_j, \tag{6·31}$$

$$D = \Sigma a_j X_j. \tag{6·32}$$

Then there results upon substitution in (6·28)

$$-\Sigma b_j(\lambda_j-\Lambda)JX_j = fJ\Sigma a_j X_j, \tag{6·33}$$

and in virtue of (6·30) it follows that

$$-b_j(\lambda_j-\Lambda) = a_j f. \tag{6·34}$$

But if one substitutes (6·31) and (6·32) into (6·23) and then employs (6·30) and (6·34) one finds

$$\frac{1}{2\pi}\left(L'+\frac{V'}{\gamma p}\right) = \Sigma\frac{a_j^2}{\Lambda-\lambda_j}. \tag{6·35}$$

The roots of (6·35) determine the possible values of Λ. Denote the right-hand side by $F(\Lambda)$ and plot it versus Λ. Note that $\mathrm{d}F/\mathrm{d}\Lambda < 0$. If none of the a_j is zero the graph is as in figure 1 and the intersections of this curve with the horizontal line $F(\Lambda) = (\frac{1}{2}\pi)(L' + V'/\gamma p)$ are the eigenvalues Λ_j of (6·28). If $a_j = 0$ for some j, the associated branch of $F(\Lambda)$ is not present in the diagram. It follows in this case from (6·34) that the associated root is $\Lambda = \lambda_j$. This is also the result which one would obtain if one considered the limit as $a_j \to 0$ of the associated intersection of the graph.

Clearly from figure 1, $\lambda_1 \leqslant \Lambda_1 \leqslant \lambda_2 \leqslant \Lambda_2 \ldots$. Thus if λ_1 is positive, so are all the Λ_j's while if λ_2 is negative, then Λ_1 is negative. If λ_1 is negative and λ_2 positive, then the sign of Λ_1 is not obvious. However, it is possible in this case to derive a criterion for the sign of Λ_1. Integrate (6·29) with respect to χ. There results

$$\lambda_j \int \mathrm{d}\chi J X_j = p' \int \mathrm{d}\chi J X_j \Sigma a_i X_i = p' a_j. \tag{6·36}$$

Thus

$$F(0) = -\Sigma \frac{a_i^2}{\lambda_i} = -\Sigma \frac{a_i}{p'} \int \mathrm{d}\chi J X_i = -\frac{1}{p'} \int \mathrm{d}\chi J D$$
$$= -\frac{1}{p'} \int \mathrm{d}\chi \left(J' - p' \frac{J}{B^2}\right) = -\frac{V''}{2\pi p'} + \frac{L'}{2\pi}. \tag{6·37}$$

Now assume that $\lambda_1 < 0$, $\lambda_2 > 0$. Since Λ_1 is determined by the condition

$$F(\Lambda_1) = \frac{1}{2\pi}(L' + V'/\gamma p)$$

and $F(\Lambda)$ is monotonically decreasing in the interval $\lambda_1 < \Lambda < \lambda_2 \ldots$, it is clear that if $F(0) > (1/2\pi)(L' + V'/\gamma p)$ then $\Lambda_1 > 0$ and conversely. But

$$F(0) - \frac{1}{2\pi}\left(L' + \frac{V'}{\gamma p}\right) = -\frac{V'}{2\pi p'}\left(\frac{V''}{V'} + \frac{p'}{\gamma p}\right). \tag{6·38}$$

One can write

$$\lambda_1 \geqslant 0 \to \Lambda_1 \geqslant 0,$$
$$\lambda_1 < 0 < \lambda_2 \to \Lambda_1 \gtreqless 0 \quad \text{as} \quad \frac{V'}{2\pi p'}\left(\frac{V''}{V'} + \frac{p'}{\gamma p}\right) \lesseqgtr 0,$$
$$\lambda_2 < 0 \to \Lambda_1 < 0. \tag{6·39}$$

In three limiting cases stability criteria can be obtained directly from (6·26), (i) if the material pressure is small compared with the magnetic pressure (i.e. $2p \ll B^2$), (ii) if the surface ψ = constant under consideration lies close to a cylinder and (iii) if the pressure gradient is large.

(a) *Case I*

Consider all quantities to be expanded in some parameter of smallness which essentially measures $2p/B^2$ and write

$$\left.\begin{aligned} p &= 0 + p^{(1)} + \ldots, & X &= X^{(0)} + X^{(1)} + \ldots, \\ B &= B^{(0)} + B^{(1)} + \ldots, & \Lambda &= \Lambda^{(0)} + \Lambda^{(1)} + \ldots, \end{aligned}\right\} \tag{6·40}$$

with similar expressions for other quantities. There results from (6·26) to lowest order,

$$\Lambda^{(0)} \int \mathrm{d}\chi J^{(0)} X^{(0)2} = \int \mathrm{d}\chi \frac{1}{[r^{(0)} B^{(0)}]^2 J^{(0)}} \left[\frac{\partial X^{(0)}}{\partial \chi}\right]^2. \tag{6·41}$$

Clearly $\Lambda^{(0)}$ is minimized by choosing $X^{(0)}$ constant in χ, which yields $\Lambda^{(0)} = 0$. This expresses the fact that the lowest order equilibrium is neither stable nor unstable, but neutral. Proceeding to the next order we find

$$\Lambda^{(1)}\int d\chi J^{(0)}X^{(0)2} = \frac{V^{(0)\prime}}{2\pi\gamma p^{(1)}}f^{(1)2} + \int d\chi p^{(1)\prime}D^{(0)}J^{(0)}X^{(0)2}, \tag{6·42}$$

which by employing (6·23) can be reduced to

$$\Lambda^{(1)} = \gamma p^{(1)}\frac{V^{(0)\prime\prime}}{V^{(0)\prime}}\left[\frac{V^{(0)\prime\prime}}{V^{(0)\prime}} + \frac{p^{(1)\prime}}{\gamma p^{(1)}}\right]. \tag{6·43}$$

The sign of $\Lambda^{(1)}$ determines stability in this case. Equation (6·43) agrees with the criterion of (6·39) in the case $2p/B^2 \ll 1$, since if $V''p' > 0$, $\lambda_1 > 0$ and both equations yield stability, while if $V''p' < 0$, $\lambda_1 < 0 < \lambda_2$ and (6·43) agrees with the second part of (6·39).

(*b*) *Case II*

Consider a surface ψ = constant. Denote by R the radius of curvature of a line of force, by L the characteristic length for the variation of equilibrium quantities along a line of force, and by a the characteristic distance in which the pressure changes by an amount comparable with itself. Assume that everywhere on this surface ψ = constant,

$$L^2 r/Ra^2 \ll 1, \tag{6·44}$$

in which circumstance the positive term in Λ proportional to $(\partial X/\partial\chi)^2$ dominates, unless $\partial X/\partial\chi = 0$ to lowest order in the parameter of smallness. Thus one is led to choose $X^{(0)}$ = constant. This leads immediately as in (6·27) to the first-order result

$$\Lambda = \gamma p(V'' - p'L')(V''/V' + p'/\gamma p)(V'' + \gamma pL')^{-1}. \tag{6·45}$$

Equation (6·45) reduces to (6·43) in the limit of small p. If $L^2r/Ra^2 \ll 1$ for all surfaces ψ = constant, then that (6·45) be negative on some surface is a necessary and sufficient condition for instability, otherwise it is only sufficient. Relation (6·44) is obviously satisfied if the surfaces are very nearly cylindrical.

Equation (6·20) gives an estimate as to the order of magnitude of R. If the two terms in the first line of (6·20) do not cancel, one obtains

$$R \sim a^2/r. \tag{6·46}$$

However, if they do cancel, as in the case of the cylinder, R is an order of magnitude larger. With this reservation, (6·44) reduces to

$$r \ll a^2/L. \tag{6·47}$$

Equation (6·45) is thus valid, for any equilibrium, for ψ surfaces close enough to the cylindrical axis.

(*c*) *Case III*

Consider an equilibrium such that everywhere on some surface ψ = constant

$$|\operatorname{grad} p| \gg B^2R/S^2, \tag{6·48}$$

where R again is the magnitude of the radius of curvature of a line of force and S is the distance over which it has the same sign. Assume that there is some region on this surface for which $p'D < 0$ and construct a trial function X which is zero outside of this region and varies smoothly within it. Then inequality (6·48) guarantees that the term in $p'D$ in (6·26) dominates and the associated Λ is less than zero. Thus the equilibrium is unstable. In the appropriate limit this case corresponds to the complete separation case of § 5.

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Some instabilities of a completely ionized plasma

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Two cases of equilibrium for a highly conducting plasma are investigated for their stability. In the first case, a plasma is supported against gravity by the pressure of a horizontal magnetic field. This equilibrium is found unstable, in close correspondence to the classical case of a heavy fluid supported by a light one. The second case refers to the so-called pinch effect. Here a plasma is kept within a cylinder by the pressure of a toroidal magnetic field which in turn is caused by an electric current within the plasma. This equilibrium is found unstable against lateral distortions.

1. Introduction

In classical hydrodynamics the problem of stability of fluid motions has been solved explicitly for a number of basic cases. Recently, Chandrasekhar (1952, 1953) has investigated and solved several of these basic problems in their hydromagnetic formulations in which electromagnetic fields are introduced and in which the fluid in question is considered electrically highly conductive. In the present paper two more cases of hydromagnetic instability are investigated.

The first case (§ 3) is that of an infinitely conducting plasma at uniform temperature lying above a horizontal plane in a uniform gravitational field directed vertically downwards. There is a horizontal magnetic field uniform in each half-volume with a jump in field strength produced by a uniform horizontal sheet current in the boundary plane. The gravitational force is balanced by a pressure gradient in the plasma and by the jump in magnetic pressure at the plane. This case is somewhat analogous to the familiar unstable equilibrium of a dense fluid supported against gravity by a lighter one (see, for instance, § 231 of Lamb 1932).

The second case (§ 4) is that of an infinitely conducting uniform plasma lying within an infinitely long circular cylinder. There is a uniform sheet current on the cylinder parallel to the axis, which produces a toroidal magnetic field outside the cylinder. There is no gravitational field. The plasma pressure is balanced by the magnetic pressure. This case is an idealization of the well-known pinch effect.

Finally, in § 5, it is investigated how far the approximation of infinite conductivity with the simultaneous introduction of sheet currents and sheet charges on the surface of the plasma is an appropriate representation of the actual cases with large but finite conductivity. This question is studied in a sample case similar to that described in § 4 but of a simpler geometry.

The results are summarized in § 6.

2. Basic equations and approximations

Let p, ρ and $\mathbf{v}$ be the pressure, density and velocity of the plasma respectively. Let $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$ and ϵ be the electric field, the magnetic field, the current density, and the electric charge density respectively. Let $\mathbf{g}$ be the acceleration due to gravity, σ the conductivity of the plasma, μ_0 the permeability of free space, κ_0 the permittivity of free space, and γ the ratio of specific heats of the plasma. We shall use the following equations for the interior of the plasma:

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{j}\times\mathbf{B}+\epsilon\mathbf{E}-\nabla p+\rho\mathbf{g}, \tag{1}$$

$$\nabla\cdot(\rho\mathbf{v}) = -\frac{\partial\rho}{\partial t}, \tag{2}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B} = \frac{1}{\sigma}(\mathbf{j}-\mathbf{v}\epsilon), \tag{3}$$

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{j}+\kappa_0\frac{\partial\mathbf{E}}{\partial t}\right), \tag{4}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{5}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{6}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\kappa_0}\epsilon, \tag{7}$$

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\gamma}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{8}$$

These equations should apply to a plasma under the following conditions. First, encounters between particles are sufficiently frequent to permit the representation of the stress tensor by a scalar pressure. Secondly, the perturbations are sufficiently slow and the electron density sufficiently high so that the extra terms which strictly should be added to equation (3) are negligible. Thirdly, heat losses by conduction and gains by Joule heating are negligible so that the simple adiabatic equation (8) holds.

In the vacuum we have $p = 0$ and $\rho = 0$, and equations (4), (5), (6) and (7) hold with $\mathbf{j} = 0$ and $\epsilon = 0$.

We next derive the equations that hold at a surface separating the plasma from the vacuum. At such a surface we allow a sheet current $\mathbf{j}^*$ and a sheet charge ϵ^*, as well as jump discontinuities in the other physical quantities. Let the unit normal to the surface be $\mathbf{n}$ (directed into the plasma), and let the surface have a normal velocity $u\mathbf{n}$. Since the surface moves with the plasma we must have

$$u = \mathbf{n}\cdot\mathbf{v}, \quad \frac{d\mathbf{n}}{dt} = \mathbf{n}\times(\mathbf{n}\times\nabla u). \tag{9}$$

The separating surface is only an approximate representation of a thin layer of plasma of width δ in which $\mathbf{j}$ and ϵ are finite but large of order δ^{-1} and across which the other physical quantities vary continuously. The integrals of $\mathbf{j}$ and ϵ across this layer are given by $\mathbf{j}^*$ and ϵ^* respectively. The integral of any one of the other quantities across the layer is of order δ and therefore vanishes with δ. If q or $\mathbf{q}$ represents any of these other quantities, then the integral of ∇q, $\nabla\cdot\mathbf{q}$, or $\nabla\times\mathbf{q}$ is given, except for terms which vanish with δ, by $\mathbf{n}[q]$, $\mathbf{n}\cdot[\mathbf{q}]$, or $\mathbf{n}\times[\mathbf{q}]$ respectively, where the brackets (in this section only) denote the difference of the quantity within them from one side of the layer to the other, or in other words, the jump of the quantity across the separating surface. Finally, we must observe that the main contribution to $\partial q/\partial t$ or $\partial\mathbf{q}/\partial t$ within the layer comes from the motion of the layer combined with the gradient of q or $\mathbf{q}$ across the layer, and is given by $-u\mathbf{n}\cdot\nabla q$ or $-u(\mathbf{n}\cdot\nabla)\mathbf{q}$ respectively.

Our procedure is now to integrate each of the plasma equations (1) to (8) across the layer and then to go to the limit as $\delta\to 0$. Equation (2) yields $\mathbf{n}\cdot[\rho\mathbf{v}] = u[\rho]$, which follows anyway from equation (9), since $\rho = 0$ in the vacuum. Equation (8) yields $0 = 0$. Equations (4), (5), (6) and (7) yield respectively

$$\mathbf{n}\times[\mathbf{B}] = \mu_0(\mathbf{j}^* - \kappa_0 u[\mathbf{E}]), \tag{10}$$

$$\mathbf{n}\cdot[\mathbf{B}] = 0, \tag{11}$$

$$\mathbf{n}\times[\mathbf{E}] = u[\mathbf{B}], \tag{12}$$

$$\mathbf{n}\cdot[\mathbf{E}] = \frac{1}{\kappa_0}\epsilon^*. \tag{13}$$

Equation (3) yields

$$\frac{1}{\sigma}(\mathbf{j}^* - \mathbf{v}\epsilon^*) = 0, \tag{14}$$

unless $\sigma = \infty$, in which case it yields $0 = 0$. Finally, equation (1) yields

$$\mathbf{j}^*\times\overline{\mathbf{B}} + \epsilon^*\overline{\mathbf{E}} - \mathbf{n}[p] = 0, \tag{15}$$

where $\overline{\mathbf{B}}$ and $\overline{\mathbf{E}}$ denote appropriate averages of the values of $\mathbf{B}$ and $\mathbf{E}$ respectively in the layer; the correct treatment of equation (1), in which $\mathbf{j}$ and ϵ are eliminated by equations (4) and (7) before the integration across the layer, shows that the proper average to take is in each case the ordinary arithmetic mean of the values on the two sides of the separating surface.

3. Instability of a plasma supported against gravity by a magnetic field

In this case we take $\sigma = \infty$, $g_x = g_z = 0$, $g_y = -g_0$. In the equilibrium let the plasma lie above the plane $y = 0$, and in it let $p = p_0 \exp(-hy)$, $\rho = \rho_0 \exp(-hy)$, $\mathbf{v} = 0$, $B_x = B_y = 0$, $B_z = B_0^P$, $\mathbf{E} = 0$, $\mathbf{j} = 0$, $\epsilon = 0$. In the vacuum let $B_x = B_y = 0$, $B_z = B_0^V$, $\mathbf{E} = 0$. On the surface $y = 0$ let $j_x^* = j_0^*$, $j_y^* = j_z^* = 0$, $\epsilon^* = 0$. Then the plasma equations (1) to (8), the vacuum equations, and the surface equations (9) to (15) are fulfilled if the constants g_0, p_0, ρ_0, h, B_0^P, B_0^V and j_0^* satisfy

$$h = g_0\rho_0/p_0, \quad B_0^P - B_0^V = \mu_0 j_0^*, \quad j_0^*(B_0^P + B_0^V) + 2p_0 = 0. \tag{16}$$

We now write a tilde over the symbol for a physical quantity to indicate its perturbation, i.e. the difference between that quantity and its value in the equilibrium solution just given; thus $\tilde{p} = p - p_0 \exp(-hy)$, $\tilde{\mathbf{v}} = \mathbf{v}$, etc. We then seek a solution near the equilibrium solution by linearizing all our equations in terms of the perturbations. The resulting homogeneous linear equations have constant coefficients except for plasma equations (1), (2) and (8), which have coefficients proportional to $\exp(-hy)$. If we restrict ourselves to investigating phenomena which in the plasma differ from the equilibrium solution appreciably only near the surface, i.e. for which the perturbations approach zero much faster than $\exp(-hy)$ as y gets large, then we may to a good approximation replace the factor $\exp(-hy)$ in the coefficients by unity.

Every solution of the resulting system of homogeneous linear differential and algebraic equations may be obtained by superposition of elementary solutions, an elementary solution being one in which each perturbation in the plasma is proportional to $\exp(\mathrm{i}k_x x - k_y^P y + \mathrm{i}k_z z + \omega t)$, each perturbation in the vacuum to $\exp(\mathrm{i}k_x x + k_y^V y + \mathrm{i}k_z z + \omega t)$, and at the surface to $\exp(\mathrm{i}k_x x + \mathrm{i}k_z z + \omega t)$. For the solution to make physical sense we require k_x and k_z to be real and k_y^P and k_y^V to have non-negative real parts, and in fact we require $\mathscr{R}[k_y^P] \gg h$ to accord with the restriction made in the previous paragraph.

For each physical quantity we denote the constant amplitude factor by the same symbol except for replacing the tilde by a circumflex. The system of differential equations for the perturbations becomes a purely algebraic system of linear equations for the amplitude factors. Introducing the dimensionless quantities

$$\left.\begin{aligned} &\beta = \mu_0 p_0/B_0^{P2}, \quad G = \beta h/k_x, \quad 1/\Gamma = 1/\gamma + \beta + p_0 k_z^2/\rho_0\omega^2, \\ &P = k_y^P/k_x, \quad V = k_y^V/k_x, \quad Z = (k_z^2 + \mu_0\kappa_0\omega^2)/k_x^2, \\ &T = (\mu_0\rho_0)^{\frac{1}{2}}\,\omega/B_0^P k_x, \end{aligned}\right\} \tag{17}$$

the conditions that the system of homogeneous equations for the amplitude factors have a non-trivial solution are found by successive elimination to be

$$\left.\begin{aligned} V^2 &= 1 + Z, \\ T^2 + Z &= PG - (1+2\beta)(\gamma P - \Gamma G)Z/\gamma V \\ &\quad + [\gamma - 1 + (\beta + 1/\gamma)\Gamma]G^2/\gamma\beta, \\ (T^2 + Z)\{P &- 1/P - [1 - (\beta + 1/\gamma)\Gamma]G + (1+2\beta)(1-\beta\Gamma)Z/V\} \\ &= [\gamma - 1 + (\beta + 1/\gamma)\Gamma]G^2/\gamma\beta P - (1+2\beta)\Gamma G Z/\gamma P V. \end{aligned}\right\} \tag{18}$$

We thus have three conditions on the five characteristic parameters k_x, k_y^P, k_y^V, k_z, ω, so we may regard k_x and k_z as given (i.e. the wave-lengths of the perturbation in the two horizontal directions as given) and the others (i.e. the extent of the perturbation upwards into the plasma and downwards into the vacuum and the characteristic time of the perturbation) to be determined. Since $\mu_0\kappa_0$ is the reciprocal of the square of the velocity of light, we have, very nearly, $Z = k_z^2/k_x^2$, so that in equations (18) we may consider G and Z as known and P, V and T to be determined; in terms of these variables

$$1/\Gamma = 1/\gamma+\beta-\mu_0\kappa_0 p_0/\rho_0+\beta Z/T^2.$$

We now assume that $|G|\ll 1$ and $|Z|\ll 1$ and seek the limiting forms of the solutions of equations (18); we obtain a solution which to lowest order is

$$|P| = 1, \quad |V| = 1, \quad T^2 = Gk_x/|k_x|-2(1+\beta)Z. \tag{19}$$

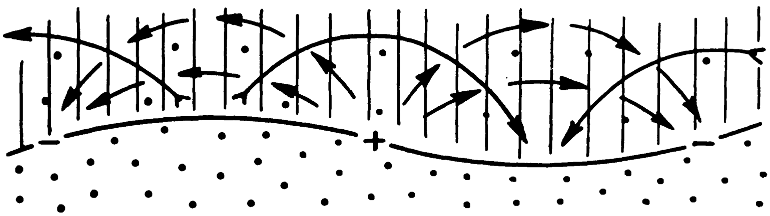

FIGURE 1. Instability of plasma supported against gravity by a magnetic field. ||| plasma, ∴ magnetic field; +, − electric charge; →→ electric field; ⤚→ motion of plasma.

Furthermore, it seems that for any other limiting solution of equations (18) T^2 becomes real and negative as G and Z approach zero, so that solution (19) represents the only possibility for instability. The condition that instability occur in this solution in the approximation of infinite light velocity is $T^2 > 0$, or

$$2(1+1/\beta)k_z^2 < h|k_x|. \tag{20}$$

Hence instability occurs if the wave-length of the perturbation along the magnetic field (z-direction) is long compared to the geometric mean of the scale height in the plasma and the wave-length across the magnetic field (x-direction).

The amplitude factors are given in table 1 for two limiting cases, one of instability when $Z = 0$, the other of stability when $G = 0$. Each amplitude factor is given to lowest order in G and in Z in the respective cases. However, terms containing the velocity of light in the denominator are omitted, except in $\hat{\mathbf{j}}$ and $\hat{\mathbf{j}}^*$, where they are retained so that the equation $\nabla\cdot\mathbf{j} = -\partial\epsilon/\partial t$ (derivable from equations (4) and (7)) can be seen to be satisfied at the boundary surface.

Finally, we get from equation (19) for the limiting case of $Z = 0$ (no dependence on z)

$$\omega^2 = g_0|k_x|, \tag{21}$$

and for the limiting case of $G = 0$ (no gravity)

$$\omega^2 = -k_z^2(B_0^{P2}+B_0^{V2})/\mu_0\rho_0. \tag{22}$$

while in both cases

$$k_y^P = k_y^V = |k_x|. \tag{23}$$

The perturbation corresponding to the first limiting case ($Z = 0$) is shown in figure 1. We may describe this perturbation approximately as follows. Electric charges on the boundary of the plasma produce electric fields. These electric fields are perpendicular to the magnetic field and hence cause the plasma to move. These motions are essentially divergence-free so that density and pressure do not vary if one follows an element of matter. The motions carry along the magnetic lines, but neither bend nor compress them, so that the magnetic field, both in the plasma and in the vacuum, stays constant. The accelerations of the plasma and the pressure gradients together produce small electric currents which tend to increase the charges on the plasma boundary, thus making the perturbation unstable. Even though in this perturbation the electromagnetic field produces the plasma motions, the speed of the instability as given by equation (21) is exactly the same as that of the well-known purely hydrodynamic case of a heavy fluid supported against gravity by a lighter one.

TABLE 1

	unstable solution $Z = 0$	stable solution $G = 0$
	plasma	
p	$p_0 h/k_x$	$-\gamma p_0(1+2\beta)k_z/(1+\gamma\beta)k_x$
ρ	$\rho_0 h/k_x$	$-\rho_0(1+2\beta)k_z/(1+\gamma\beta)k_x$
v_x	$-i\omega/k_x$	$-i\omega/k_z$
v_y	ω/k_x	ω/k_z
v_z	0	0
B_x	0	B_0^P
B_y	0	iB_0^P
B_z	0	$-B_0^P(1+2\beta)k_z/(1+\gamma\beta)k_x$
E_x	$-B_0^P\omega/k_x$	$-B_0^P\omega/k_z$
E_y	$-iB_0^P\omega/k_x$	$-iB_0^P\omega/k_z$
E_z	0	0
j_x	$[1+(1+2\beta)/(1+\gamma\beta)]\kappa_0 g_0 B_0^P k_x/\lvert k_x\rvert$	$[1+(1+2\beta)/(1+\gamma\beta)]B_0^P k_z/\mu_0$
j_y	$i[1+(1+2\beta)/(1+\gamma\beta)]\kappa_0 g_0 B_0^P$	$i[1+(1+2\beta)/(1+\gamma\beta)]B_0^P k_z/\mu_0$
j_z	0	0
ϵ	0	0
	vacuum	
B_x	0	$-B_0^V$
B_y	0	iB_0^V
B_z	0	$-B_0^V k_z/k_x$
E_x	$-B_0^V\omega/k_x$	$-B_0^V\omega/k_z$
E_y	$iB_0^V\omega/k_x$	$iB_0^V\omega/k_z$
E_z	0	0
	boundary surface	
j_x^*	$\kappa_0 g_0[B_0^V - B_0^P(1+2\beta)/(1+\gamma\beta)]/k_x$	$[B_0^V - B_0^P(1+2\beta)/(1+\gamma\beta)][1-2(1+1/\beta)$ $\times\mu_0\kappa_0 p_0/\rho_0]k_z/\mu_0 k_x$
j_y^*	$i(B_0^P - B_0^V)/\mu_0$	$i(B_0^P - B_0^V)k_x/\mu_0 k_z$
j_z^*	0	$-(B_0^P + B_0^V)/\mu_0$
ϵ^*	$-i\kappa_0(B_0^P + B_0^V)\omega/k_x$	$-i\kappa_0(B_0^P + B_0^V)\omega/k_z$

The second limiting case ($G = 0$) represents a hydromagnetic surface wave which travels on the boundary of the plasma along the magnetic lines with the characteristic speed given by equation (22).

The discriminating condition (20) between the stable and the unstable cases may be described as follows. If the wave-length along the magnetic lines is sufficiently short, the restoring force of the magnetic field which resists the bending of the magnetic lines will prevent their sagging. On the other hand, if the wave-length of the perturbation along the magnetic lines is too long, the perturbation will bend the lines only a little and the magnetic restoring force will be too small to counteract gravity, so that the plasma will drop downwards.

4. Lateral instability of plasma cylinder in toroidal magnetic field

In this case we take $\sigma = \infty$, $\mathbf{g} = 0$, and use cylindrical co-ordinates r, θ, z. In the equilibrium let the plasma lie within the cylinder $r = r_0$, and in it let $p = p_0$, $\rho = \rho_0$, and $\mathbf{v}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{j}$ and ϵ vanish. In the vacuum let $B_r = B_z = 0$, $B_\theta = B_0 r_0/r$, $\mathbf{E} = 0$. On the surface $r = r_0$ let $j_r^* = j_\theta^* = 0$, $j_z^* = j_0^*$, $\epsilon^* = 0$. Here r_0, p_0, ρ_0, B_0 and j_0^* are constants, and we require

$$B_0 = \mu_0 j_0^*, \quad j_0^* B_0 = 2p_0, \tag{24}$$

so that the equations of § 2 be satisfied.

We now investigate solutions in the neighbourhood of this equilibrium solution as before by indicating the perturbations from the equilibrium by a tilde. In the equations of § 2 we retain only the first powers of these perturbations and obtain a system of homogeneous differential and algebraic equations with coefficients depending only on r. Accordingly, the elementary solutions of this system have the form $\tilde{q} = \hat{q}\exp(\mathrm{i}m\theta + \mathrm{i}kz + \omega t)$, where q is any physical quantity, $\hat{q}$ is a function of r (unless q is a surface quantity), m is an integer, and k is real. We restrict ourselves to the case $m = 1$.

Substitution gives a system of ordinary differential equations for the amplitude factors which reduce, after eliminations, to

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{p}\right) - [(k^2 + \rho_0\omega^2/\gamma p_0)r^2 + 1]\hat{p} = 0 \tag{25}$$

in the plasma, and to

$$\left.\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{B}_z\right) - [(k^2 + \mu_0\kappa_0\omega^2)r^2 + 1]\hat{B}_z = 0, \\ r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\hat{E}_z\right) - [(k^2 + \mu_0\kappa_0\omega^2)r^2 + 1]\hat{E}_z = 0 \end{aligned}\right| \tag{26}$$

in the vacuum. We introduce the functions

$$J(x) = -\mathrm{i}J_1(\mathrm{i}x), \quad H(x) = -H_1^{(1)}(\mathrm{i}x), \tag{27}$$

where J_1 and $H_1^{(1)}$ are the Bessel and Hankel functions of the first kind and of first order. We further introduce the constants

$$\zeta^2 = k^2 + \rho_0\omega^2/\gamma p_0, \quad \eta^2 = k^2 + \mu_0\kappa_0\omega^2. \tag{28}$$

Assuming that ω^2 is real and positive and taking $\zeta \geqslant 0$, $\eta \geqslant 0$, we see from equations (25) and (26) that

$$\hat{p} = p^1 J(\zeta r), \quad \hat{B}_z = B_z^1 H(\eta r), \quad \hat{E}_z = E_z^1 H(\eta r), \tag{29}$$

where p^1, B_z^1, E_z^1 are constants. The other independent solutions of equations (25) and (26) are excluded because they become infinite at $r = 0$ and $r = \infty$ respectively. The surface equations now give three independent linear homogeneous relations among p^1, B_z^1 and E_z^1, and the condition that these have a non-trivial solution is

$$\frac{\rho_0 r_0^2 \eta \omega^2}{2p_0 \zeta} \frac{J(\zeta r_0)}{J'(\zeta r_0)} = \eta r_0 + \frac{k^2}{\eta^2} \frac{H(\eta r_0)}{H'(\eta r_0)} + \mu_0 \kappa_0 r_0^2 \omega^2 \frac{H'(\eta r_0)}{H(\eta r_0)}, \tag{30}$$

TABLE 2

	exact solution in terms of	approximate solution for
	$B_z^1 = \dfrac{p_0 B_0 k \zeta}{\rho_0 r_0 \omega^2 \eta} \dfrac{J'(\zeta r_0)}{H'(\eta r_0)}$	$\mu_0 \kappa_0 \approx 0$ and $\lvert k \rvert r_0 \ll 1$
	$E_z^1 = \dfrac{p_0 B_0 \zeta}{\rho_0 \omega} \dfrac{J'(\zeta r_0)}{H(\eta r_0)}$	
	plasma	
p	$p_0 J(\zeta r)$	$\frac{1}{2} \lvert k \rvert p_0[1 + 2L/\gamma]^{\frac{1}{2}} r$
ρ	$\rho_0 J(\zeta r)/\gamma$	$\frac{1}{2} \lvert k \rvert \rho_0[1 + 2L/\gamma]^{\frac{1}{2}} r/\gamma$
v_r	$-p_0 \zeta J'(\zeta r)/\rho_0 \omega$	$-\frac{1}{2}[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}$
v_θ	$-\mathrm{i} p_0 J(\zeta r)/\rho_0 \omega r$	$-\frac{1}{2}\mathrm{i}[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}$
v_z	$-\mathrm{i} k p_0 J(\zeta r)/\rho_0 \omega$	$-\frac{1}{2}\mathrm{i}k[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} r$
	vacuum	
B_r	$-\mathrm{i}\mu_0 \kappa_0 \omega E_z^1 H(\eta r)/\eta^2 r - \mathrm{i} k B_z^1 H'(\eta r)/\eta$	$-\frac{1}{4}\mathrm{i} B_0 r_0[1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r^2$
B_θ	$k B_z^1 H(\eta r)/\eta^2 r + \mu_0 \kappa_0 \omega E_z^1 H'(\eta r)/\eta$	$-\frac{1}{4} B_0 r_0[1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r^2$
B_z	$B_z^1 H(\eta r)$	$-\frac{1}{4} k B_0 r_0[1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L r$
E_r	$\mathrm{i}\omega B_z^1 H(\eta r)/\eta^2 r - \mathrm{i} k E_z^1 H'(\eta r)/\eta$	$\frac{1}{2}\mathrm{i} k B_0 r_0[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 + L + \ln(r_0/r)]$
E_θ	$k E_z^1 H(\eta r)/\eta^2 r - \omega B_z^1 H'(\eta r)/\eta$	$\frac{1}{2} k B_0 r_0[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 - L - \ln(r_0/r)]$
E_z	$E_z^1 H(\eta r)$	$\frac{1}{2} B_0 r_0[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}}/r$
	boundary surface	
j_r^*	$-\mathrm{i} k p_0 B_0 \zeta J'(\zeta r_0)/\mu_0 \rho_0 \omega^2$	$-\frac{1}{4}\mathrm{i} k j_0^*[1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L$
j_θ^*	$-B_z^1 H(\eta r_0)/\mu_0$	$\frac{1}{4} k j_0^*[1 + 2L/\gamma]^{\frac{1}{2}}/\lvert k \rvert L$
j_z^*	$k B_z^1 H(\eta r_0)/\mu_0 \eta^2 r_0 + \kappa_0 \omega E_z^1 H'(\eta r_0)/\eta + p_0 B_0 \zeta J'(\zeta r_0)/\mu_0 \rho_0 r_0 \omega^2$	$\frac{1}{4} \lvert k \rvert j_0^* r_0[1 + 2L/\gamma]^{\frac{1}{2}}$
ϵ^*	$\mathrm{i}\kappa_0 \omega B_z^1 H(\eta r_0)/\eta^2 r_0 - \mathrm{i} k \kappa_0 E_z^1 H'(\eta r_0)/\eta$	$\frac{1}{2}\mathrm{i} k \kappa_0 B_0 r_0[p_0(1/L + 2/\gamma)/2\rho_0]^{\frac{1}{2}} \times [L r_0^2/r^2 + L + \ln(r_0/r)]$

where J' and H' are the derivatives of J and H. If we normalize by taking $p^1 = p_0$, then B_z^1 and E_z^1 are determined by these relations. Their values are given in table 2 at the head of a column. The remainder of this column contains the exact solution expressed in terms of B_z^1 and E_z^1; $\hat{\mathbf{B}}$, $\hat{\mathbf{E}}$, $\hat{\mathbf{j}}$ and $\hat{\epsilon}$ all vanish in the plasma and are therefore not listed in the table.

We now neglect terms which have the light velocity in the denominator, such as the last term of the second equation (28) and the last term of equation (30). Thus, equations (28) give $\eta = \lvert k \rvert$ and equation (30) gives

$$\frac{\rho_0 r_0^2 \lvert k \rvert \omega^2}{2p_0 \zeta} \frac{J(\zeta r_0)}{J'(\zeta r_0)} = \lvert k \rvert r_0 + \frac{H(\lvert k \rvert r_0)}{H'(\lvert k \rvert r_0)}. \tag{31}$$

It can be proved that, for any fixed value of $k \neq 0$, the left-hand side of this equation varies monotonically from 0 to ∞ as ω^2 goes from 0 to ∞ through real values; since the right-hand side can be shown to be positive, it follows that equation (31) has a unique real positive solution ω^2 for each value of k. Thus the equilibrium is unstable. This instability is in agreement with the earlier results of Lundquist (1951).

In the limiting case in which the wave-length of the perturbation along the cylinder is long compared to the cylinder radius, i.e. in which $|k| r_0 \ll 1$, equation (31) gives for the main term in ω (taken positive)

$$\omega \approx |k| (2Lp_0/\rho_0)^{\frac{1}{2}} \quad \text{with} \quad L = \ln \frac{2}{|k| r_0} - C + 1, \tag{32}$$

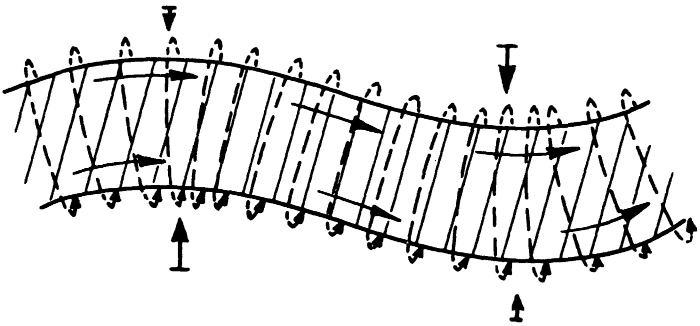

FIGURE 2. Lateral instability of plasma cylinder in toroidal magnetic field. /// plasma; → electric current; ↺ magnetic field; ⊥ magnetic force.

where $C \simeq 0{\cdot}5772$ is Euler's constant. The amplitude factors for this limiting case are given in the last column of table 2. In the opposite limiting case of large $|k| r_0$, equation (31) gives

$$\omega \approx (2|k| p_0/\rho_0 r_0)^{\frac{1}{2}}. \tag{33}$$

Equations (32) and (33) show that these unstable perturbations e-fold in about the time that it takes a sound wave to travel a distance equal to the wave-length of the perturbation or equal to the geometric mean of the wave-length and the cylinder radius, whichever is the larger.

The character of this unstable perturbation of a plasma cylinder is shown in figure 2. (This figure actually shows a lateral sinusoidal perturbation obtained by superposing two oppositely twisted helical perturbations, i.e. two perturbations as given in table 2 with k's of opposite sign.) The cause of the instability can be described as follows. When the plasma cylinder is distorted into the wavy form shown in figure 2, the toroidal magnetic lines are separated from each other at the convex side of the cylinder and pressed together at the concave side. In consequence, the magnetic pressure which holds the cylinder together is put out of balance, being weakened on the convex side and strengthened on the concave side. The net result is that the total magnetic force tends to increase the perturbation and thus leads to instability.

Another problem, similar to the one detailed above, was computed through in which the plasma instead of being confined by a cylindrical surface was confined by two parallel planes. In the equilibrium configuration, the plasma was again kept in its boundary by the pressure of a magnetic field, which was parallel to the planes and larger outside the plasma than inside. In contrast with the foregoing cylindrical case, it was found that this plane case is neutral to perturbations in which the two confining surfaces are distorted into sinusoidal wave-forms. Furthermore, for this simple case, it was easy to compute through the perturbations for two different assumptions regarding the current distribution in the equilibrium configuration. In the first version it was assumed that the equilibrium current was entirely confined to the surface of the plasma—similar to the assumption in the cylindrical case—and in the second version it was assumed that the equilibrium current was evenly distributed throughout the plasma between the two planes. In both versions the same neutrality of equilibrium was found. This latter result may be taken to indicate that the stability question for the type of equilibria here considered is not seriously affected by the form of the assumed current distribution.

5. Effects of finite conductivity

In this case we take σ finite but very large, $\mathbf{g} = 0$. Let the plasma lie between the two planes $x = \pm x_0$, and in it let $p = p_0(1-x^2/x_0^2)$, $\rho = \rho_0(1-x^2/x_0^2)$, $B_y = B_0 x/x_0$, $E_z = E_0$, $j_z = j_0$ and $\mathbf{v}$, ϵ and the other components of $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{j}$ be zero. In the vacuum let $B_x = B_z = 0$, $B_y = B_0$ for $x > x_0$, $B_y = -B_0$ for $x < -x_0$, $\mathbf{E} = 0$. On the surfaces $x = \pm x_0$ let $\mathbf{j}^* = 0$, $\epsilon^* = 0$. We then have an equilibrium system if the constants x_0, p_0, ρ_0, B_0, E_0, j_0 and σ satisfy

$$E_0 = j_0/\sigma, \quad B_0 = \mu_0 j_0 x_0, \quad j_0 B_0 x_0 = 2p_0. \tag{34}$$

We now introduce the perturbations as usual and seek solutions of the somewhat restricted form $\tilde{q} = \hat{q}\exp(ikz+\omega t)$, where the amplitude factor $\hat{q}$ is a function of x and k is real. The resulting system of equations for the amplitude factors splits into two entirely separate systems, one for $\hat{v}_y$, $\hat{B}_x$, $\hat{B}_z$, $\hat{E}_y$, $\hat{j}_y$ and $\hat{j}_y^*$, the other for the remaining variables. We confine our attention to the latter system.

We introduce the dimensionless constants and functions

$$\left.\begin{aligned} \alpha &= \mu_0\kappa_0 p_0/\rho_0, & w^2 &= \rho_0 x_0^2\omega^2/\gamma p_0, \\ \lambda &= \tfrac{1}{2}(\alpha^{-1}+2)\mu_0 x_0^2\sigma\omega, & \nu^2 &= (2\gamma^{-1}+1)\mu_0 x_0^2\sigma\omega, \\ \eta^2 &= x_0^2(k^2+\mu_0\kappa_0\omega^2), & R &= 1+2\gamma^{-1}x_0^{-2}x^2, \\ S &= 1-2(1-\gamma^{-1})x_0^{-2}x^2, & T &= 1+2\alpha x_0^{-2}x^2, \end{aligned}\right\} \tag{35}$$

choosing $\mathscr{R}[\nu] \geqslant 0$ and $\mathscr{R}[\eta] \geqslant 0$.

Now any solution for the amplitude factors can be written as the sum of an odd and an even solution, where an odd solution is one in which the functions $\hat{v}_x$, $\hat{B}_y$, $\hat{E}_x$, $\hat{j}_x$ are all odd in x, and the functions $\hat{p}$, $\hat{\rho}$, $\hat{v}_z$, $\hat{E}_z$, $\hat{j}_z$, $\hat{\epsilon}$ are all even in x, and vice versa for even solutions. For a rough comparison with the cylindrical case, odd solutions correspond to radial perturbations and even solutions to perturbations varying as $\exp(i\theta)$, which we have discussed in the preceding section. Accordingly, we shall restrict ourselves here to even solutions.

The system of differential (and algebraic) equations for the amplitude factors in the plasma is of fifth order and is singular at $x = 0$. It has three independent even solutions.

We are interested in the asymptotic solutions as $\sigma \to \infty$. We assume that ω has a limit different from zero. If we set $\sigma = \infty$ the system of equations in the plasma reduces to a non-singular system of second order, having one even solution. This solution is given in the first column of table 3, where ϕ is a function of x vanishing at $x = 0$ and satisfying

$$x_0^2\phi'' + 2x(R^{-1} - 2\alpha T^{-1})\,\phi' = (k^2x_0^2 + w^2R^{-1}T)\,\phi, \tag{36}$$

the prime denoting differentiation with respect to x.

TABLE 3

	column 1	column 2 (each entry to be multiplied by $e^{\nu(x-x_0)/x_0}$)
p	$p_0R^{-1}(\gamma\phi - 2w^{-2}ST^{-1}x\phi')$	$2p_0$
ρ	$\rho_0R^{-1}(\phi - 2w^{-2}T^{-1}x\phi')$	$2\rho_0/\gamma$
v_x	$-\gamma p_0(\rho_0\omega T)^{-1}\,\phi'$	$-2x_0\omega/\gamma\nu$
v_z	$-ik\gamma p_0(\rho_0\omega T)^{-1}\,\phi$	$-2ikx_0^2\omega/\gamma\nu^2$
B_y	$B_0(x_0R)^{-1}\,(x\phi + x_0^2w^{-2}ST^{-1}\phi')$	$-B_0$
E_x	$-ik\gamma p_0B_0(\rho_0x_0\omega T)^{-1}\,x\phi$	$ikx_0^2B_0\omega/\nu^2$
E_z	$\gamma p_0B_0(\rho_0x_0\omega T)^{-1}\,x\phi'$	$-x_0B_0\omega/\nu$
j_x	$-ikj_0[(R^{-1} - \gamma\alpha T^{-1})\,x\phi + x_0^2w^{-2}R^{-1}ST^{-1}\phi']$	ikx_0j_0
j_z	$j_0[(S+2)\,R^{-1} + 1 + k^2x_0^2w^{-2}ST^{-1}]\,\phi$ $-j_0w^{-2}T^{-1}[2(S+2)\,R^{-1} - (1-\gamma\alpha)\,w^{-2}]\,x\phi'$	$-j_0\nu$
ϵ	$ik\gamma\alpha j_0(\omega T)^{-1}\,(1 - 2T^{-1})\,\phi$	$-2ikj_0(1+\gamma\alpha)\,w^2/(1/\alpha + 2)\,\nu^2$

	column 3 (each entry to be multiplied by $e^{\lambda(x-x_0)/x_0}$)	column 4 (each entry to be multiplied by $e^{-\eta(x-x_0)/x_0}$)
p	$ikx_0p_0[\gamma(1+2\alpha)\,w^2 + 4(2/\gamma - 1)/(1/\alpha + 2)]/\lambda^2$	
ρ	$-2ikx_0\rho_0(1 - 1/\gamma)/\lambda$	
v_x	$-ikx_0^2\omega/\lambda$	
v_z	$x_0\omega$	
B_y	$-2ikx_0B_0(2/\gamma - 1)/(1/\alpha + 2)\,\lambda^2$	$-B_0\gamma\alpha w^2$
E_x	$x_0B_0\omega$	$ikx_0^2B_0\omega$
E_z	$ikx_0^2B_0\omega/\lambda$	$x_0B_0\omega\eta$
j_x	$-j_0\gamma\alpha w^2$	
j_z	$-ikx_0j_0\alpha[\gamma w^2 + 2(2/\gamma - 1)/(1+2\alpha)]/\lambda$	
ϵ	$(j_0/x_0\omega)\,\gamma\alpha w^2\lambda$	

The asymptotic forms of the two remaining even solutions are not obtained by this procedure, since they do not converge uniformly as $\sigma \to \infty$. Instead, they drop off from their values at $x = x_0$ more and more rapidly as σ gets larger and larger. If we rewrite our system of equations in terms of the new independent variable $\xi = (x - x_0)\,\sigma^{\frac{1}{2}}$ and then let $\sigma \to \infty$, the system becomes asymptotically a non-singular system of fourth order. This has one even solution in addition to an asymptotically constant (in ξ) solution corresponding to the solution already found (given in the first column). Written as a function of x, the main term in each amplitude factor of this solution is given in the second column of table 3.

If we write the system of equations in terms of the new independent variable $\zeta = (x - x_0)\,\sigma$ (instead of the variable ξ above) and then let $\sigma \to \infty$, the system becomes asymptotically a non-singular system of fifth order. This has a third even solution in addition to two even asymptotically constant (in ζ) solutions corresponding to the two even solutions already found (given in the first two columns of table 3). Written as a function of x, the main term in each amplitude factor of this solution is given in the third column of table 3.

So much for the equations in the plasma. As for the equations in the vacuum, they are easily solved exactly, the amplitude factors being given in the fourth column of table 3. (The similar mathematical solution with exponent of opposite sign becomes infinite as $x \to \infty$ and is therefore excluded.)

The surface equations give $\hat{u} = \hat{v}_x$, $\hat{n}_z = -ik\omega^{-1}\hat{v}_x$, $\hat{j}^*_x = \hat{j}^*_z = 0$, $\hat{\epsilon}^* = 0$, and the four joining conditions

$$\hat{p} - \frac{2p_0}{x_0\omega}\hat{v}_x = 0, \quad \hat{E}^P_x = \hat{E}^V_x, \quad \hat{E}^P_z = \hat{E}^V_z, \quad \hat{B}^P_y + \frac{B_0}{x_0\omega}\hat{v}_x = \hat{B}^V_y, \tag{37}$$

where all the amplitude factors are to be evaluated at $x = x_0$, the superscripts P and V distinguishing between plasma and vacuum quantities respectively. Now, the solution in the plasma is a linear combination of the independent solutions given in the first, second and third columns of table 3, say with coefficients a_1, a_2 and a_3 respectively, and the solution in the vacuum is a multiple of the solution given in the fourth column, say by a_4. Equations (37) thus give four linear homogeneous equations for a_1, a_2, a_3 and a_4. Asymptotically as $\sigma \to \infty$ the condition that these have a non-trivial solution becomes

$$\phi + \frac{2\alpha}{1+2\alpha}\frac{x_0}{\eta}\phi' = 0, \tag{38}$$

and the solution itself becomes

$$\left.\begin{aligned} a_2 &= -(2\gamma^{-1}+1)^{-1}(1+2\alpha)^{-1}(2w^{-2}-\gamma\alpha\eta^{-1})x_0\phi' a_1, \\ a_3 &= ikx_0^2(1+2\alpha)^{-2}w^{-2}\eta^{-1}\phi' a_1, \\ a_4 &= (1+2\alpha)^{-1}w^{-2}\eta^{-1}x_0\phi' a_1; \end{aligned}\right\} \tag{39}$$

in equations (38) and (39) ϕ and ϕ' are to be evaluated at $x = x_0$.

The characteristic time constants ω are determined by the eigenvalues w^2 of the differential equation (36) subject to the boundary conditions $\phi = 0$ at $x = 0$ and (38) at $x = x_0$. Any such eigenvalue is real and negative; in fact, it is easily seen that $w^2 < -k^2x_0^2$. Thus ω is purely imaginary and the elementary solutions we have found are oscillations of constant amplitude.

We are interested in these solutions only as examples of perturbations in plasmas with large but finite σ. According to equations (35), ν is a positive real multiple of $(1+i)\sigma^{\frac{1}{2}}$ and λ of $\pm i\sigma$. Hence, the second particular solution listed in table 3 decreases exponentially from the plasma boundary inwards, and the more rapidly the larger σ is. If our solution were carried through to the next order terms in σ, presumably λ would contain a positive real multiple of $\sigma^{\frac{1}{2}}$, so that the third particular solution listed in table 3 would drop off similarly away from the plasma boundary. Thus, the second and third solutions may be taken as describing skin phenomena. Indeed, in the limit of $\sigma \to \infty$, the only quantities in these two particular solutions which remain of physical interest are j_z of the second solution, which contains a factor ν and hence converges to a finite sheet current equal to $-j_0x_0$, and ϵ of the third solution which contains a factor λ and hence converges to a sheet charge equal to $\gamma\alpha w^2 j_0/\omega$. It may be mentioned that this whole case was also computed through with $\sigma = \infty$ from the start, and the solutions were entirely what would be expected as the limits of the solutions of the general case, including sheet quantities of just the right magnitudes.

This example of a perturbation in a plasma with finite but large conductivity seems therefore to indicate that the essential features of the type of problems here considered are fairly represented if one assumes infinite conductivity but simultaneously allows for electric sheet currents and electric sheet charges at the surface of the plasma.

6. Summary

In this paper, two types of instabilities have been discussed for highly conductive plasmas in magnetic fields. The first type of instability arises in a plasma which is supported against gravity by magnetic pressure. Such a plasma is found to be unstable against perturbations which move the magnetic lines essentially parallel to themselves but do not bend them seriously. This instability is found to be of the same type and of the same speed of development as the well-known instability of a heavy fluid supported against gravity by a lighter one.

The second instability arises in the well-known pinch effect, i.e. for a plasma which is contained within a cylinder by toroidal magnetic fields which in turn are caused by electric currents within the plasma parallel to the cylinder axis. Such a plasma is unstable against perturbations which distort the cylinder into a sinusoidal tube (figure 2). The e-folding time of this perturbation is roughly equal to the time a sound wave travels a distance equal to the wave-length of the perturbation.

These two instabilities are computed through under the approximation of infinite conductivity, with proper allowance for electric currents and charges on the surface of the plasma. The belief that this approximation is in fact adequate has been strengthened by investigating a particularly simple case in which some sample perturbations could be computed through under the assumption of large but finite conductivity.

We are happy to acknowledge our indebtedness to Dr Lyman Spitzer, Jr, for his constant advice and in particular for suggesting the possible existence of the instabilities here investigated.

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I

Finite-Resistivity Instabilities of a Sheet Pinch

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The stability of a plane current layer is analyzed in the hydromagnetic approximation, allowing for finite isotropic resistivity. The effect of a small layer curvature is simulated by a gravitational field. In an incompressible fluid, there can be three basic types of "resistive" instability: a long-wave "tearing" mode, corresponding to breakup of the layer along current-flow lines; a short-wave "rippling" mode, due to the flow of current across the resistivity gradients of the layer; and a low-g gravitational interchange mode that grows in spite of finite magnetic shear. The time scale is set by the resistive diffusion time τ_R and the hydromagnetic transit time τ_H of the layer. For large $S = \tau_R/\tau_H$, the growth rate of the "tearing" and "rippling" modes is of order $\tau_R^{-3/5}\tau_H^{-2/5}$, and that of the gravitational mode is of order $\tau_R^{-1/3}\tau_H^{-2/3}$. As $S \to \infty$, the gravitational effect dominates and may be used to stabilize the two nongravitational modes. If the zero-order configuration is in equilibrium, there are no overstable modes in the incompressible case. Allowance for plasma compressibility somewhat modifies the "rippling" and gravitational modes, and may permit overstable modes to appear. The existence of overstable modes depends also on increasingly large *zero-order* resistivity gradients as $S \to \infty$. The three unstable modes merely require increasingly large gradients of the *first-order* fluid velocity; but even so, the hydromagnetic approximation breaks down as $S \to \infty$. Allowance for isotropic viscosity increases the effective mass density of the fluid, and the growth rates of the "tearing" and "rippling" modes then scale as $\tau_R^{-2/3}\tau_H^{-1/3}$. In plasmas, allowance for thermal conductivity suppresses the "rippling" mode at moderately high values of S. The "tearing" mode can be stabilized by conducting walls. The transition from the low-g "resistive" gravitational mode to the familiar high-g infinite conductivity mode is examined. The extension of the stability analysis to cylindrical geometry is discussed. The relevance of the theory to the results of various plasma experiments is pointed out. A nonhydromagnetic treatment will be needed to achieve rigorous correspondence to the experimental conditions.

I. INTRODUCTION

A PRINCIPAL result of pinch[1,2] and stellarator[3] research has been the observed instability of configurations that the hydromagnetic theory[4,5] would predict to be stable in the limit of high electrical conductivity. In order to establish the cause of this observed instability, the extension of the hydromagnetic analysis to the case of finite conductivity becomes of considerable interest.

A number of particular "resistive" instability modes have been discussed in previous publications. Dungey[6] has shown that, at an x-type neutral point of a magnetic-field structure in plasma, finite conductivity can give rise to an unstably growing current concentration. By Dungey's mechanism, a sheet pinch can tear along current-flow lines, so as

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to form discrete parallel filaments.[7,8] This "tearing" mode is purely growing and is symmetric about the midplane of the sheet pinch.

Murty[9] has analyzed the case of a very-low-conductivity incompressible fluid slab of finite thickness, and has found two purely growing modes: the symmetric "tearing" mode; and an asymmetric "rippling" mode. In the latter case, the conductivity gradient at the edge of the slab permits current channeling into first-order "ripples" that run at an angle with respect to both the zero-order current and the zero-order magnetic field. The resultant motor force amplifies the ripples.

Aitken *et al.*[2,10] have treated cylindrical geometry, and have found a purely growing (helical) "rippling" mode in the very-low-conductivity limit. In the high-conductivity limit, they find an overstable "rippling" mode.[2,11] The ripples in the latter case run in the direction of the mean zero-order current. The existence of overstability depends on the compressibility of the fluid and on large resistivity gradients.

The instability of the positive column[12,13] is somewhat related to the instability of fully ionized plasmas of finite conductivity. Kadomtsev and Nedospasov[14] have demonstrated a "rippling" mode, which is purely growing in the rest frame of the electrons, but is overstable in the laboratory frame. The extension of this mode to fully ionized plasmas has been considered by Hoh,[15] Kuckes,[16] and Kadomtsev.[17]

In the present analysis, general equations are derived for the plane resistive current layer in the incompressible hydromagnetic approximation. A dispersion relation is obtained in the limit of high conductivity that describes purely growing modes of the "tearing" and "rippling" types. An interchange mode driven by a gravitational field perpendicular to the plane layer is also included.

The analysis for the plane current layer is particularly significant in the high-conductivity limit, since the problem then separates into the analysis of two regions: (1) a narrow central region, where finite conductivity permits relative motions of field and fluid, and where geometric curvature may be neglected; (2) an outer region, where field and fluid are coupled as in the infinite-conductivity case, and where generalizations to nonplanar geometry can be introduced as desired.

In Sec. II the problem is delineated and the basic assumptions and equations displayed. In Secs. III–V a formal mathematical solution is developed. In Sec. VI the basic physical mechanisms are discussed and a simple heuristic derivation and summary of the results is given. For those not interested in mathematical details or preferring a preliminary physical discussion, it is suggested that Sec. VI be read prior to Secs. III–V. Section VII is devoted to a comparison with experiment. The effects of various generalizations and extensions of the basic problem are considered in the appendixes as follows: Appendix A. Compressibility; Appendix B. Low-Conductivity Limit; Appendix C. Short Wavelength; Appendix D. Long-Wavelength Limit; Appendix E. The Transition to the ∞-Conductivity Limit of the Rayleigh–Taylor Instability; Appendix F. Thermal Conductivity; Appendix G. External Conductors; Appendix H. Viscosity; Appendix I. Cylindrical Geometry.

II. ASSUMPTIONS AND BASIC EQUATIONS

We treat an infinite plane current layer specified by

$$\mathbf{B}_0 = \hat{x}B_{x0}(y) + \hat{z}B_{z0}(y) \tag{1}$$

The following assumptions are made.

1. The hydromagnetic approximation is assumed to be valid, and the ion pressure and inertia terms are neglected in Ohm's law.

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times[(\eta/4\pi)\nabla\times\mathbf{B}]. \tag{2}$$

As the analysis will show, these assumptions are violated in the treatment of a plasma of sufficiently high conductivity, since the "resistive" modes then develop increasingly sharp discontinuities, and we must expect "finite-Larmor-radius" effects. Plasma stability in the limit of high but finite conductivity (the limit of maximum practical interest) thus depends critically on nonhydromagnetic effects. An isotropic resistivity η is assumed in Eq. (2), and the mass of the electrons is neglected. It is of interest to note that inclusion of the electron-inertia term in Ohm's law gives rise to a "tearing" mode in

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the collisionless limit[18] that is analogous to the "resistive tearing" mode considered here.

2. The fluid is assumed to be incompressible.

$$\nabla \cdot \mathbf{v} = 0. \quad (3)$$

In the high-conductivity limit the effect of compressibility on the fluid dynamics is negligible. (See Appendix A.)

3. Viscosity is neglected, so that the equation of motion may be written as

$$\nabla \times (\rho\, d\mathbf{v}/dt) = \nabla \times [(1/4\pi)(\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{g}\rho] \quad (4)$$

where ρ is the mass density and $\mathbf{g}$ the acceleration due to gravity. As usual, the $\mathbf{g}\rho$ term may be interpreted as resulting from acceleration of the current layer, or from the interaction of a plasma pressure gradient and a slight curvature of the current layer.[19] The effect of viscosity is discussed in Appendix H.

4. Perturbations in plasma resistivity are assumed to result only from convection.

$$\partial\eta/\partial t + \mathbf{v} \cdot \nabla\eta = 0. \quad (5)$$

The neglect of thermal conductivity along magnetic field lines, however, becomes important for a high-temperature plasma,[17] and we must then use the equation

$$\frac{\partial\eta}{\partial t} + \mathbf{v} \cdot \nabla\eta = \frac{-\eta}{nT}\,\mathbf{B} \cdot \nabla\left(\frac{\kappa \mathbf{B} \cdot \nabla T}{B^2}\right), \quad (5a)$$

where κ is the coefficient of thermal conductivity along magnetic field, n is the particle density, and T the temperature. The associated stabilizing effect against the "rippling" mode is discussed in Appendix F. The neglect of Ohmic heating in Eq. (5) is unimportant in the high-conductivity short-wavelength limit. For low-conductivity plasma, a small amount of Ohmic heating due to first-order currents tends to accelerate the "rippling" mode and retard the "tearing" mode. If the Ohmic heating is sufficiently strong to *reduce* the local electric field at a current concentration, a trivial type of "tearing" instability results that depends primarily on thermal rather than fluid transport effects. The effect of plasma compressibility on Eq. (5) is discussed in Appendix A.

5. Perturbations in $\mathbf{g}\rho$ are assumed to result only from convection

$$\partial(\mathbf{g}\rho)/\partial t + \mathbf{v} \cdot \nabla(\mathbf{g}\rho) = 0. \quad (6)$$

In the presence of a neutral background gas, the first-order currents may, however, give rise to an additional density perturbation by an increase in the local rate of ionization. This mechanism may have a destabilizing effect.[20,21] The effect of compressibility is discussed in Appendix A.

6. The zero-order distribution will be assumed to have $\mathbf{v}_0 = 0$. Strictly speaking, this condition implies

$$\nabla \times (\eta_0 \nabla \times \mathbf{B}_0) = 0 \quad (7)$$

which will be referred to as the standard case. For modes with sufficiently large growth rates and wavelengths, however, the approximation of null zero-order velocity is valid even if Eq. (7) is not strictly satisfied, since the values of $\mathbf{v}_0$ to be expected are those of ordinary resistive diffusion. Some of our results will therefore be presented in their most general form, without invoking Eq. (7).

Denoting perturbed quantities by the subscript 1,

$$f_1(\mathbf{r}, t) = f_1(y) \exp[i(k_x x + k_z z) + \omega t]$$

we obtain to first order the set of equations

$$\omega \mathbf{B}_1 = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) - (1/4\pi)\nabla \times [\eta_0 \nabla \times \mathbf{B}_1 + \eta_1 \nabla \times \mathbf{B}_0], \quad (8)$$

$$\omega \nabla \times \rho_0 \mathbf{v}_1 = \nabla \times \{(1/4\pi)[(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 + (\mathbf{B}_1 \cdot \nabla)\mathbf{B}_0] + (\mathbf{g}\rho)_1\}, \quad (9)$$

$$\nabla \cdot \mathbf{v}_1 = \nabla \cdot \mathbf{B}_1 = 0, \quad (10)$$

$$\omega\eta_1 + (\mathbf{v}_1 \cdot \nabla)\eta_0 = 0, \quad (11)$$

$$\omega(\mathbf{g}\rho)_1 + (\mathbf{v}_1 \cdot \nabla)(\mathbf{g}\rho)_0 = 0. \quad (12)$$

From this set of equations, we may separate two that involve only B_{y1} and v_{y1}. Equations governing the remaining first-order quantities (not needed in the present analysis) are given in Appendix A. In dimensionless form, we have

$$\frac{\psi''}{\alpha^2} = \psi\left(1 + \frac{p}{\bar{\eta}\alpha^2}\right) + \frac{W}{\alpha^2}\left(\frac{F}{\bar{\eta}} + \frac{\bar{\eta}'F'}{\bar{\eta}p}\right), \quad (13)$$

$$\frac{(\bar{\rho}W')'}{\alpha^2} = W\left[\bar{\rho} - \frac{S^2G}{p^2} + \frac{FS^2}{p}\left(\frac{F}{\bar{\eta}} + \frac{\bar{\eta}'F'}{\bar{\eta}p}\right)\right] + \psi S^2\left(\frac{F}{\bar{\eta}} - \frac{F''}{p}\right), \quad (14)$$

where

$$\psi = B_{y1}/B, \qquad W = -iv_{y1}k\tau_R,$$

$$F = (k_x B_{x0} + k_z B_{z0})/kB, \qquad k = (k_x^2 + k_z^2)^{\frac{1}{2}},$$

$$\alpha = ka, \qquad \tau_R = 4\pi a^2/\langle\eta\rangle, \qquad \tau_H = a(4\pi\langle\rho\rangle)^{\frac{1}{2}}/B,$$

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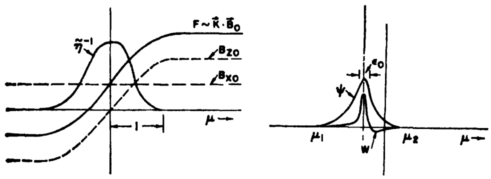

FIG. 1(a) Equilibrium configuration of sheet pinch. (b) Form of the perturbation.

$$S = \tau_R/\tau_H, \quad p = \omega\tau_R, \quad \tilde{\eta} = \eta_0/\langle\eta\rangle,$$
$$\tilde{\rho} = \rho_0/\langle\rho\rangle, \quad G = \tau_H^2 A_1.$$

The primes denote differentiation with respect to a dimensionless variable $\mu = y/a$, where a is a measure of the thickness of the current layer. The quantities B, $\langle\eta\rangle$, and $\langle\rho\rangle$ are measures of the field strength, resistivity, and mass density respectively. The quantity A_1 may be interpreted as $-(g/\rho_0)\ \partial\rho_0/\partial y$ for the gravitational-field case; or as $-(1/\rho_0)\ \partial(\rho_0\dot{v}_0)/\partial y$ for a current layer with zero-order acceleration $\dot{v}_0$; or very roughly as $-(1/\tau_H^2)(a^2/4R_c)\ \partial\beta_0/\partial y$ for a current-layer of mean radius of magnetic curvature R_c and plasma pressure P_0, where $\beta_0 = 4\pi P_0/B^2$. The latter application is discussed in detail in Appendix I.

For thermal plasmas, we have approximately $S \sim 0.1aT^2\beta_0^{-\frac{1}{2}}$, with T in eV. The parameter S exceeds a hundred for most present-day hot-plasma experiments and must become much larger yet in experiments of thermonuclear interest. Accordingly, our primary interest will be in the case $S \to \infty$. Note that the growth rate p is expressed in units of the resistive diffusion time.

Only one component of the $\mathbf{B}_0$ field, namely $\mathbf{k}(\mathbf{k}\cdot\mathbf{B}_0)$ appears in Eqs. (13) and (14). For any given $\mathbf{B}_0$ field having finite shear, we may choose $\mathbf{k}$ so that $\mathbf{k}\cdot\mathbf{B}_0 \sim F$ passes through a null. The typical μ-dependence of F and η that will be considered here is illustrated in Fig. 1(a).

The zero-order equilibrium condition of Eq. (7) may be written as

$$(\tilde{\eta}'/\tilde{\eta})F' = -F''. \tag{15}$$

The usual boundary conditions are that both ψ and W should vanish at infinity or at conducting boundaries, located at $\mu = \mu_1, \mu_2$.

III. GENERAL REMARKS

Nonexistence of Overstable Modes

Equations (13) and (14) can be solved in the limit $S \to \infty$, to give an oscillatory mode [Re $(p) = 0$]. Expanding about this solution in powers of S^{-1}, one finds that, when Im $(p) \neq 0$, either Re $(p) = O(1)$ (so that the growth rate is insignificant); or else the zero-order current layer must have sharp resistivity gradients, which become increasingly so as $S \to \infty$. In a more general analysis, including plasma compressibility, etc., we would expect the same result. This follows since the equations can always be expanded in powers of S^{-1}, as long as the zero-order conductivity is large everywhere and has finite gradients. Thus the modes of greatest practical interest are new modes that do not exist at all for $S = \infty$. The situation is similar to that for hydrodynamic shear-flow stability at high Reynolds number.[22]

In the incompressible case, we can show that no overstable modes exist at all, provided that non-equilibrium zero-order configurations are excluded by requiring Eq. (7) (or 15) to be satisfied. This condition is appropriate for configurations with sharp resistivity gradients, since the zero-order diffusion velocity could not otherwise be neglected. For convenience, we will use a definition of the quantity $\langle\eta\rangle$ such that $\tilde{\eta}F' = 1$. Equations (13), (14) can then be rewritten in the form

$$\frac{p^2}{\alpha^2 S^2 F}\left[(\tilde{\rho}W')' + \alpha^2 W\left(\frac{S^2 G}{p^2} - \tilde{\rho}\right)\right]$$
$$= (p\psi + WF)\left(pF' - \frac{F''}{F}\right) \tag{16}$$
$$= p\psi'' - p\psi\left(\alpha^2 + \frac{F''}{F}\right). \tag{17}$$

Equations (16) and (17) yield the condition

$$\int_{\mu_1}^{\mu_2} d\mu \left\{\frac{p^2}{|p|^2\alpha^2 S^2}\left[\tilde{\rho}\ |W'|^2 + \alpha^2\ |W|^2\left(\tilde{\rho} - \frac{S^2 G}{p^2}\right)\right] + \frac{pF' - F''/F}{|pF' - F''/F|^2}\left|\psi'' - \psi\left(\alpha^2 + \frac{F''}{F}\right)\right|^2 + |\psi'|^2 + |\psi|^2\left(\alpha^2 + \frac{F''}{F}\right)\right\} = 0. \tag{18}$$

Taking the imaginary part of Eq. (18), we find that if Im $(p) \neq 0$, then Re $(p) \leq 0$.

Characteristics of Unstable Modes

We will devote primary attention to those unstable modes for which $S \to \infty$ and $p \sim S^{\zeta}$ where $0 < \zeta < 1$. The lower limit on ζ corresponds to a growth rate that is of the same order as the rate of resistive diffusion, and is therefore insignificant. The upper limit on ζ is reached only by modes that exist also in the standard infinite-conductivity treatment.

Since the growth rates of the modes to be considered are slow compared with the hydromagnetic rates, the flow is subsonic; i.e., the incompressibility approximation is satisfactory (cf. Appendix A). On the other hand, since the growth rates are fast compared with resistive diffusion rates, the effect of Ohmic heating is negligible.

A discussion of unstable modes in the limit $S \to 0$ is given in Appendix B. It is shown that in this limit the growth rates approach hydromagnetic rates.

For unstable modes, all quantities in Eq. (18) are real. In the limit $S \to \infty$, Eq. (18) can be satisfied in three distinct ways, each corresponding to a negative contribution from one of the three terms: (1) if $G > 0$, there can be gravitationally driven modes; (2) if ψ is peaked near the point $F = 0$, and if we can have $F''/F > 0$ at this point (i.e., if $\eta' \neq 0$ there), then there are modes corresponding to the "rippling" instability; (3) since F''/F is predominantly negative, for sufficiently small α^2 there are modes corresponding to the "tearing" instability.

The behavior of the solutions over most of the range in μ can be established on a general basis. As $S \to \infty$, we must have

$$p\psi \approx -FW \tag{19}$$

everywhere except in a small interval near $F = 0$. This condition follows from the consideration that either W or ψ would diverge strongly at large μ if the right-hand term in Eq. (16) were either negative or positive except in a small interval. Eq. (19) is, of course, the condition that the fluid remains "frozen" to magnetic field lines.

Using Eq. (19), we then see from Eqs. (16) and (17) that the (infinite-conductivity) equation

$$\psi'' - \psi(\alpha^2 + F''/F - G/F^2) = 0 \tag{20}$$

must be satisfied everywhere except in a small interval. The general procedure in the $S \to \infty$, $0 < \zeta < 1$ limit is therefore as follows. We obtain solutions to Eq. (20) that vanish at $\mu = \mu_1, \mu_2$, the external boundaries. These solutions cannot, in general, be joined without a discontinuity in ψ',

$$\Delta' = \psi_2'/\psi_2 - \psi_1'/\psi_1, \tag{21}$$

where the subscripts refer to values on either side of the point of juncture. The typical behavior of ψ is illustrated in Fig. 1(b). The discontinuity in ψ'/ψ corresponds to large local values of ψ''. From Eqs. (16) and (17) we see that such values can be obtained only near the point $F = 0$. Equation (13) implies that large local values of W are also obtained near the same point and only there. The second stage of the general solution therefore consists in solving for ψ and W in a small region R_0 about the point $F = 0$, with the boundary conditions that ψ'/ψ matches the solutions of Eq. (20), and that W is well behaved outside the region R_0.

In more formal terms, we may say that Eqs. (19) and (20) provide an asymptotic solution of Eqs. (16) and (17), which breaks down near $F = 0$. We note that if $F \neq 0$ everywhere, then Eq. (20) applies throughout, and there is no solution unless $G/(F')^2 = O(1)$, in which case the layer is unstable even in the $S = \infty$ limit.

The argument of this section has, for reasons of convenience, made use of Eqs. (16) and (17), which refer specifically to the standard case [i.e., Eq. (15) holds]. The conclusions can, however, be extended to more general choices of F, if desired.

IV. SOLUTIONS IN THE OUTER REGION

We assume that Eq. (20) holds everywhere outside a small region R_0 with a width of order ϵ_0 around the point μ_0 at which $F = 0$. Eq. (20) is to be solved subject to the boundary condition $\psi = 0$ at the points μ_1, μ_2, which we will take for convenience at $\mp\infty$. We will calculate the quantity Δ' of Eq. (21) for the case $\psi_1 = \psi_2$ which is of principal interest in Sec. V. Equation (20) yields the expression

$$\Delta' = -2\alpha - \frac{1}{\psi_1}\int_{-\infty}^{\infty} d\mu\, e^{-\alpha|\mu|}\psi\,\frac{F''}{F} + O\left[\frac{G}{(F')^2\epsilon_0}\right]. \tag{22}$$

Note that when $F'' \neq 0$ at μ_0, there is a singularity in the integrand on the right side of Eq. (22). Difficulties arising from the corresponding logarithmic singularity in ψ' are avoided here, since we consider only Δ' instead of the individual values of ψ_1' and ψ_2'. In this and the following section we will restrict ourselves to the case where $|G|/(F')^2$ is sufficiently small so that the G term in Eq. (22) can be neglected. The case of larger G is discussed in Appendix E.

For the case $\alpha^2 \gg 1$, one obtains from Eq. (22)

$$\Delta' = -2\alpha + O(1/\alpha). \tag{23}$$

For the case $\alpha^2 \ll 1$, we expand

$$\psi = e^{-\alpha|\mu-\mu_0|}(\psi_{(0)} + \alpha\psi_{(1)} + \alpha^2\psi_{(2)} \cdots)$$

and find

$$-(\psi_{(n)}F' - \psi_{(n)}'F)' = [2|\mu - \mu_0|/(\mu - \mu_0)]F\psi_{(n-1)}'.$$

The well-behaved solution is characterized by

$$\psi_{(0)} = |F| \tag{24}$$

and near the point μ_0 by

$$\psi_{(1)} = F^2_{-\infty}/F', \ \mu < \mu_0; \ \psi_{(1)} = F^2_{\infty}/F', \ \mu > \mu_0. \tag{25}$$

In calculating Δ', the derivative of $\psi_{(1)}$ may be neglected, though it has a logarithmic singularity at μ_0, since the contributions it makes near μ_0 cancel out, and the rest is of order α. Thus we obtain

$$\Delta' = (1/\alpha)(F')^2(1/F^2_{-\infty} + 1/F^2_{\infty}). \tag{26}$$

In the case of symmetric F''/F, it will be of interest to obtain Δ' for arbitrary α. For this purpose it is convenient to choose specific models. When

$$F = \tanh \mu, \tag{27}$$

then Eq. (20) may be solved explicitly in terms of associated Legendre functions, and we have

$$\Delta' = 2(1/\alpha - \alpha). \tag{28}$$

When

$$F = \mu, \ |\mu| < 1; \quad F = 1, \ \mu > 1; \quad F = -1, \quad \mu < -1; \tag{29}$$

we have

$$\Delta' = 2\alpha\left[\frac{(1-\alpha) - \alpha \tanh \alpha}{\alpha - (1-\alpha)\tanh \alpha}\right]. \tag{30}$$

Note that Δ' goes monotonically from ∞ to $-\infty$ as α goes from 0 to ∞. There is a null of Δ' at the point $\alpha = \alpha_c$, which occurs at 1 and 0.64 respectively for the models of Eqs. (27) and (29).

V. SOLUTIONS IN THE REGION OF DISCONTINUITY

Basic Equations

In the small region R_0 about the point μ_0 we may take the quantities F', F'', $\bar{\eta}$, $\bar{\eta}'$, G, and $\bar{\rho}$ in Eqs. (13) and (14) to be constant. We may approximate F as $F'(\mu - \mu_0)$ and neglect the term $\bar{\rho}'W'$ relative to $\bar{\rho}W''$.

Defining a new independent variable

$$\theta = (1/\epsilon)(\mu - \mu_0 + \bar{\eta}'/2p), \tag{31}$$

Eqs. (13) and (14) may be written as

$$d^2\psi/d\theta^2 - \epsilon^2\alpha^2\psi = \epsilon\Omega[4\psi + U(\theta + \delta_1)], \tag{32}$$

$$d^2U/d\theta^2 + U(\Lambda - \tfrac{1}{4}\theta^2) = \psi(\theta - \delta), \tag{33}$$

where

$$\epsilon = [p\bar{\eta}\bar{\rho}/4\alpha^2S^2(F')^2]^{\frac{1}{4}}, \tag{34}$$

$$U = W(4\epsilon F'/p), \tag{35}$$

$$\Omega = p\epsilon/4\bar{\eta}, \tag{36}$$

$$\delta = (1/4\Omega)(F''/F' + \bar{\eta}'/2\bar{\eta}), \tag{37}$$

$$\delta_1 = (1/8\Omega)(\bar{\eta}'/\bar{\eta}), \tag{38}$$

$$\Lambda = (\bar{\eta}')^2/16\epsilon^2p^2 + S^2\alpha^2\epsilon^2G/p^2\bar{\rho} - \alpha^2\epsilon^2. \tag{39}$$

Note that in the standard case [cf. Eq. (15)] we have $\delta = -\delta_1$.

Let us expand U in terms of normalized Hermite functions

$$U = \sum_{n=0}^{\infty} a_n u_n, \tag{40}$$

where

$$d^2u_n/d\theta^2 + (n + \tfrac{1}{2} - \tfrac{1}{4}\theta^2)u_n = 0 \tag{41}$$

and

$$u_n = \frac{(-1)^n}{(2\pi)^{\frac{1}{4}}(n!)^{\frac{1}{2}}} e^{\frac{1}{4}\theta^2} \frac{d^n}{d\theta^n} e^{-\frac{1}{2}\theta^2}. \tag{42}$$

Then Eq. (33) can be written in the form

$$a_n = \frac{1}{\Lambda - (n + \frac{1}{2})} \int_{-\infty}^{\infty} d\theta_1 \, u_n\psi(\theta_1 - \delta). \tag{43}$$

Equations (32) and (33) are valid over the range $(\mu - \mu_0)^2 \ll 1$. We will apply these equations over a region R_0 of width ϵ_0, outside of which Eqs. (19) and (20) are to be valid. From Eqs. (32) and (33) it follows then that we must require $\epsilon_0 > \epsilon$.

The most important class of unstable modes corresponds to the approximation ψ = const in R_0. For this case we obtain the "tearing" and "rippling" modes over a range of α consistent with $\epsilon_0 \,|\psi'/\psi| < 1$ in R_0, or roughly

$$\epsilon_0 \,|\Delta'| < 1. \tag{44}$$

Using the requirement $\epsilon_0 > \epsilon$ and the results of Eqs. (23) and (26), we may rewrite Eq. (44) as

$$\epsilon(F')^2(1/F^2_{-\infty} + 1/F^2_{\infty}) < \alpha < 1/2\epsilon \tag{44a}$$

For ψ = const in R_0 we also obtain the low-G gravitational interchange mode. Sufficient conditions for the constancy of ψ and for the negligibility of the G term in Eq. (20) are provided by Eq. (44a) together with the requirement that $|G|(F')^{-2}$ be small compared with ϵ, or compared with $\epsilon\,|\Delta'|$ when $|\Delta'| \gg 1$. In Appendix E, we will show that the condition on G can in general be relaxed considerably. We may take ψ to be constant over $\epsilon_0 = (1 + |\Delta'|)^{-1}$. Then the G term in Eq. (20) will be small (when $G > 0$), provided that we have

$$G/(F')^2 < \tfrac{1}{4} \tag{45}$$

and

$$G/(F_{*\infty})^2 < \alpha^2. \tag{45a}$$

We note that Eq. (45) is equivalent to the well-known Suydam criterion for instability of an infinite-conductivity plasma at short wavelengths.

Solutions with Constant ψ

If ψ = const and Eqs. (44) and (45) are satisfied, we obtain from Eqs. (32) and (43)

$$\Delta' = \Omega \sum_{n=0}^{\infty} 4\left[\int_{-\infty}^{\infty} d\theta_1\, u_n\right]^2 + \frac{1}{\Lambda - (n + \frac{1}{2})}$$
$$\cdot\left[\int_{-\infty}^{\infty} d\theta_1\, u_n(\theta_1 + \delta_1)\right]\left[\int_{-\infty}^{\infty} d\theta_2\, u_n(\theta_2 - \delta)\right]. \tag{46}$$

Using the integrals

$$\int_{-\infty}^{\infty} d\theta_1\, u_n = 2^{\frac{1}{4}}\left[\frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}n + 1)}\right]^{\frac{1}{2}} \quad (n \text{ even});$$
$$= 0 \quad (n \text{ odd});$$
$$\int_{-\infty}^{\infty} d\theta_1\, \theta_1 u_n = 0 \quad (n \text{ even}),$$
$$= 2^{9/4}\left[\frac{\Gamma(\frac{1}{2}n + 1)}{\Gamma(\frac{1}{2}n + \frac{1}{2})}\right]^{\frac{1}{2}} \quad (n \text{ odd});$$

we obtain the eigenvalue equation

$$\Delta' = 2^{7/2}\Omega \sum_{m=0}^{\infty} \frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 1)}$$
$$\cdot\left[\frac{\Lambda - \frac{1}{2}}{\Lambda - (2m + \frac{3}{2})} - \frac{\delta\delta_1/4}{\Lambda - (2m + \frac{1}{2})}\right], \tag{47}$$

where Δ' is determined by the "outside" solutions (cf. Sec. IV). We have replaced even n by $2m$ and odd n by $2m + 1$. The series is convergent, since terms for large m go like $m^{-\frac{3}{2}}$. [Note, however, that if we had calculated $\psi' |_0^\infty$, we would have obtained a divergent result, proportional to

$$(\delta_1 - \delta) \sum_{m=0}^{\infty} \frac{1}{m}.$$

Since $(\delta_1 - \delta) \sim F''/F'$, and $\theta^2 \sim m$, we may identify this divergence with the logarithmic singularity indicated by Eq. (22).]

The sums can be evaluated as hypergeometric series of argument 1 to give

$$\Delta' = 2^{7/2}\pi\Omega\left[\frac{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)} + \frac{\delta\delta_1}{8}\frac{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}\right]. \tag{47a}$$

A form that is sometimes more convenient is

$$\Delta' \pm \left[(\Delta')^2 - \pi^2\frac{\bar{\eta}'}{\bar{\eta}}\left(\frac{2F''}{F'} + \frac{\bar{\eta}'}{\bar{\eta}}\right)\right]^{\frac{1}{2}}$$
$$= \pi 2^{9/2}\Omega\frac{\Gamma(\frac{3}{4} - \frac{1}{2}\Lambda)}{\Gamma(\frac{1}{4} - \frac{1}{2}\Lambda)}. \tag{47b}$$

The following general remarks can be made about the solutions of Eq. (47).

1. If $\delta\delta_1 < 0$ (the most common case), then Δ'/Ω goes from ∞ to $-\infty$ as Λ goes from $\frac{1}{2}$ to $\frac{3}{2}$, from $\frac{3}{2}$ to $\frac{5}{2}$, etc. The quantity Ω, related to Λ by Eqs. (36) and (39), is finite for finite Λ. Hence for any given Δ', as obtained from Eq. (21), there is an infinite sequence of eigenvalues $\Lambda \sim 1, 2, 3, \cdots$. There is also an eigenvalue below $\frac{1}{2}$, which moves to 0 as $\Delta' \to \infty$, while Ω becomes large.

2. If $\delta\delta_1 = 0$ then Δ'/Ω goes from ∞ to $-\infty$ as Λ goes from $2m + \frac{3}{2}$ to $2m + \frac{7}{2}$. The sequence of eigenvalues is $\Lambda \sim 2, 4, 6 \cdots$. There is also an eigenvalue below $\frac{3}{2}$.

3. If $0 < \delta\delta_1 \ll \Lambda$ then Δ'/Ω covers almost the entire range from ∞ to $-\infty$ as Λ goes from $2m + \frac{3}{2}$ to $2m + \frac{7}{2}$. The excluded interval is

$$|\Delta'/\Omega| < 4\pi(\delta\delta_1)^{\frac{1}{2}}. \tag{48}$$

For Δ'/Ω outside the excluded interval, there is also an eigenvalue below $\frac{3}{2}$.

4. When $|\Lambda| \ll \frac{1}{2}$, Eq. (47) reduces to

$$\Delta' = \Omega(12 + 13\,\delta\delta_1). \tag{49}$$

In that case Ω is to be determined by the value of Δ' in Eq. (21), and the condition on Λ is to be verified by means of Eqs. (36) and (39). Evidently this case arises only for positive Δ'.

We can now identify a number of basic modes.

The "Rippling" Mode

The "rippling" mode is characterized by the finiteness of Λ and the predominance of the $(\eta')^2$ term on the right side of Eq. (39). In that case

$$p = \left[\frac{(\bar{\eta}')^2\alpha S\,|F'|}{8\Lambda\bar{\eta}^{\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}}\right]^{2/5}, \tag{50}$$

$$\Omega = |\bar{\eta}'|/16\bar{\eta}\Lambda^{\frac{1}{2}}, \tag{51}$$

$$\delta_1 = 2\Lambda^{\frac{1}{2}}, \tag{52}$$

$$\epsilon = |\bar{\eta}'|/4p\Lambda^{\frac{1}{2}}. \tag{53}$$

In the standard case we have $\delta\delta_1 = -4\Lambda$, and the remarks made above in paragraph 1 then apply to the Λ spectrum. (For other reasonable choices of $\delta\delta_1 < 0$, the Λ spectrum is modified only slightly.)

In the limit $\alpha \gg 1$, which according to Eq. (23) corresponds to large negative Δ', we find that the eigenvalues Λ lie slightly below the points $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$. For the fastest growing mode, which corresponds to a solution U that is basically symmetric near μ_0, we have $\Lambda \approx \frac{1}{2}$. As we move towards the other limit, $\alpha \ll 1$, (i.e., large positive Δ') the eigenvalues that were slightly below $\frac{3}{2}$, $\frac{5}{2}$, $\cdots$ move to points slightly above $\frac{1}{2}$, $\frac{3}{2}$, $\cdots$. The fastest growing mode of this series again occurs for $\Lambda \approx \frac{1}{2}$, and corresponds to a basically symmetric U in the neighborhood of μ_0. From Eq. (50) we see that the growth rates of these modes become small as $\alpha \to 0$. The eigenvalue lying below $\frac{1}{2}$ moves toward 0 as $\alpha \to 0$. This mode goes over into the "tearing" mode (see below); the associated U becomes antisymmetric, and the growth rate becomes large as $\alpha \to 0$.

If we depart from the standard case and consider the limit $\delta = 0$, $\delta_1 \neq 0$ (cf. paragraph 2), the eigenvalues near $\frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$, $\cdots$ disappear. If $\delta\delta_1 > 0$ (cf. paragraph 3), there is no solution for $|\Delta'| \ll 1$, but for large or small α there are eigenvalues near $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$.

Using Eqs. (50) and (53), we may express the condition of Eq. (44) as

$$\left[\frac{\bar{\eta}^6\bar{\rho}(F')^8}{16S^2\Lambda^{\frac{1}{2}}(\bar{\eta}')^4}\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)^5\right]^{1/7} < \alpha < \left[\frac{S^2(F')^2\Lambda^{\frac{1}{2}}}{2|\bar{\eta}'|\bar{\eta}\bar{\rho}}\right]^{\frac{1}{2}}.$$

The behavior of the "rippling" mode for larger α is discussed in Appendix C.

As we have noted in Sec. II, paragraph 4, the use of Eq. (5) to give the first-order resistivity becomes inaccurate for a high-temperature plasma, where thermal conductivity along magnetic field lines is highly effective. For the "rippling" mode, which depends critically on the nature of the resistivity perturbation, the growth rate is then actually much smaller than would be indicated by Eq. (50). An estimate of the correction factor is given in Appendix F.

The stabilization of the "rippling" mode by gravitational effects is discussed in the section on the gravitational mode.

The "Tearing" Mode

The "tearing" mode is characterized by the condition $|\Lambda| \ll \frac{1}{2}$ (cf. paragraph 4). Eq. (49) shows that this mode is limited to positive Δ', i.e., to $\alpha \leq \alpha_c \sim 1$ [cf. Eqs. (28) and (30)]. The growth rate is obtained from Eqs. (34) and (36):

$$p = 4(\alpha S\Omega^2\bar{\eta}^{\frac{3}{2}}\bar{\rho}^{-\frac{1}{2}}|F'|)^{2/5}; \qquad (54)$$

and the condition on Λ can be expressed by means of Eq. (39):

$$\Lambda = \frac{1}{\Omega^2}\left[\left(\frac{\bar{\eta}'}{16\bar{\eta}}\right)^2 + \frac{pG}{64\bar{\eta}(F')^2}\right] \ll 1. \qquad (55)$$

For $\alpha \ll 1$, we have from Eqs. (26) and (49)

$$\Omega = \frac{1}{12\alpha}(F')^2\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right) \qquad (56)$$

so that

$$p = (F')^2\left(\frac{2S\bar{\eta}^{\frac{1}{2}}}{9\alpha\bar{\rho}^{\frac{1}{2}}}\right)^{2/5}\left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)^{4/5}. \qquad (57)$$

The fastest growing mode is generally obtained for the "symmetric case" where $F'' = 0$ at $F = 0$. A lower limit to α is set by Eq. (44)

$$\alpha > \left(\frac{1}{F_{-\infty}^2}+\frac{1}{F_{\infty}^2}\right)\left(\frac{\bar{\eta}\bar{\rho}^{\frac{1}{2}}|F'|^9}{3.3S}\right)^{\frac{1}{4}}. \qquad (58)$$

The maximal growth rate p_m thus goes as $S^{\frac{1}{2}}$. (Appendix D treats this limit by a method that avoids the constant $-\psi$ approximation but confirms the present result for p_m.)

If the current layer is perfectly symmetric, so that the nulls of $\bar{\eta}'$ and G occur at the same point $\mu = 0$, Eq. (55) is always satisfied for a mode where $\mu_0 = 0$. More generally, we see that the $(\eta')^2$ term in Eq. (55) is always negligible for $\alpha \ll 1$. The effect of the gravitational term when $G \neq 0$ at μ_0 is discussed in the next section.

For modes of the "tearing" type, the solution U is basically antisymmetric, since the $\bar{\eta}'$ terms can usually be neglected by symmetry or because $\alpha \ll 1$.

The Gravitational Interchange Mode

The gravitational interchange mode is characterized by the finiteness of Λ and the predominance of the G term on the right side of Eq. (39). Instability is obtained for $G > 0$, and the appropriate growth rate is

$$p = (S\alpha G\bar{\eta}^{\frac{1}{2}}/2\Lambda\,|F'|\,\bar{\rho}^{\frac{1}{2}})^{\frac{1}{2}}. \qquad (59)$$

The magnitude of G for which Eq. (59) holds is restricted by Eqs. (45) and (45a).

To evaluate Λ, we note from Eqs. (36) and (39) that Ω is given by

$$\Omega = G/16\epsilon\Lambda(F')^2.$$

For $\delta\delta_1 < 0$, Eq. (47) gives a series of eigenvalues Λ lying in the intervals $\frac{1}{2}$ to $\frac{3}{2}$, $\frac{3}{2}$ to $\frac{5}{2}$, etc. There may also be an eigenvalue in the interval 0 to $\frac{1}{2}$, if Δ'/Ω is not too large. Unless $G(F')^{-2}$ is of order ϵ or less, the quantity Ω is generally very large, so that $|\Delta'|/\Omega$, $|\delta\delta_1| \ll 1$. In that case the eigenvalues lie at $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc.

If $\delta\delta_1 = 0$, there is a series of eigenvalues lying in the intervals $\frac{3}{2}$ to $\frac{7}{2}$, $\frac{7}{2}$ to $\frac{11}{2}$, etc., and there may also be an eigenvalue in the interval 0 to $\frac{3}{2}$, if Δ'/Ω is not too large. For $|\Delta'|/\Omega \ll 1$, we have $\Lambda = \frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$, etc. These eigenvalues are obtained for example, for pure gravitational modes with $G(F')^{-2}$ finite as $S \to \infty$.

We now consider the effect of the G term on the growth of the "rippling" mode. The condition that the resistivity-gradient term should be dominant on the right side of Eq. (39) is

$$p < (\bar{\eta}')^2(F')^2/4\bar{\eta}\ |G| \tag{60}$$

so that, unless $G = 0$, the gravitational term always predominates in the limit $S \to \infty$, $p \to \infty$. If $G > 0$, then instability continues above the limit set by Eq. (60) in the form of gravitational modes or mixed "gravitational-rippling" modes. If $G < 0$, there are no gravitationally driven modes, and Eq. (60) then sets an upper limit to the growth of all short-wave interchange instabilities.

We next consider the effect of the G-term on the growth of the "tearing" mode, neglecting the resistivity gradient terms (cf. the preceding section). Using Eqs. (34), (36), and (39) we find that

$$p = 4^3\Lambda\Omega^2\bar{\eta}(F')^2/G. \tag{61}$$

If $G > 0$, then the "tearing" mode, which is characterized by $\Lambda \ll \frac{1}{2}$ and by the consequent applicability of Eq. (49), is restricted by the condition

$$p < 2\bar{\eta}(\Delta')^2(F')^2/9G. \tag{62}$$

As $S \to \infty$, $p \to \infty$, Eq. (62) is violated, Λ moves up toward $\frac{3}{2}$, and we have the gravitational or mixed "gravitational-tearing" mode. If $G < 0$, then Λ is negative, and Eqs. (47) and (61) yield the condition

$$p < \bar{\eta}(\Delta')^2(F')^2/17\ |G| \tag{62a}$$

in the limit $S \to \infty$. Eq. (62a) then sets an upper limit to the growth of long-wave instabilities. [We note that in the present analysis the quantity $G(F')^{-2}$ is limited by Eq. (45a), so that for extremely small α Eq. (62a) is not very restrictive.]

We may summarize the gravitational effects qualitatively in terms of four characteristic ranges of G.

I. $G < 0$. The gravitational force is stabilizing. For finite $|G|(F')^{-2}$ essentially all the resistive instabilities are suppressed in the limit $S \to \infty$; i.e., we have $p \sim S^0$ for the "rippling" and "tearing" modes.

II. $G = 0$, or at least $GS^{2/5} < 1$. In this case we may have the pure "rippling" or "tearing" modes with $p \sim S^{2/5}$.

III. $G > 0$, but not large enough for infinite-conductivity instabilities. In this case we have $p \sim S^{\frac{1}{3}}$.

IV. $G > 0$ and large enough for infinite-conductivity instabilities; i.e., $G(F')^{-2} > \frac{1}{4}$ for short-wave modes. In that case, of course, $p \sim S$.

VI. SUMMARY AND ELUCIDATION OF PRINCIPAL RESULTS

In the high-S limit, a current layer with finite gradients has three basic unstable modes and no overstable modes. The approximate properties of the unstable modes in their characteristic parametric range are summarized in Table I. Here it has been assumed that the dimensionless quantities F', F'', etc., are all of order unity. References are given to the more exact equations of the main text, and to supplementary material that more clearly defines the range of validity of the analysis and extends it somewhat. We will now discuss and rederive the modes of Table I in heuristic terms.

The existence of the three "resistive" instabilities depends on the local relaxation of the constraint that fluid must remain attached to magnetic field. For a zero-order field that is not a vacuum field, possibilities of lowering potential energy are always present; the introduction of finite conductivity makes some energetically possible modes topologically accessible. In the case of the infinite-conductivity modes, lines of force that are initially distinct must remain so during the perturbation. For the three "resistive" modes, lines of force that are initially distinct link up during the perturbation. These modes have no counterpart in the infinite-conductivity limit and disappear altogether, their

TABLE I. Summary of approximate properties of unstable modes in the high-S limit.

Mode	Range of Instability	Growth Rate p	Region of Disc. ϵ	Relevant Equations	Valid Range of Equations	Supplementary Equations
"Rippling"	$\bar{\eta}' \neq 0$	$\alpha^{2/5}S^{2/5}$	$\alpha^{-2/5}S^{-2/5}$	(50)–(53)	$S^{-2/7} < \alpha < S^{2/3}$ $\|G\| < \alpha^{-2/5}S^{-2/5}$	(60) Appendixes A, C, F, H
"Tearing"	$\alpha < 1$	$\alpha^{-2/5}S^{2/5}$	$\alpha^{-3/5}S^{-2/5}$	(54)–(58)	$S^{-1/4} < \alpha$ $\|G\| < \alpha^{-8/5}S^{-2/5}$	(62) Appendixes D, G, H, I
Gravitational Interchange	$G > 0$	$\alpha^{2/3}S^{1/3}G^{2/3}$	$\alpha^{-1/3}S^{-1/3}G^{1/6}$	(59)	$S^{-1/4}G^{1/8} < \alpha < S^{1/2}G^{-1/4}$ $G < 1, \alpha^2$ $G > \alpha^{-2/5}S^{-2/5}, \alpha^{-8/5}S^{-2/5}$	(60)–(62) Appendixes A, C, E, H, I

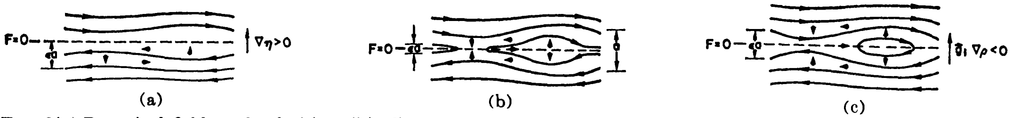

FIG. 2(a) Perturbed fields and velocities—"rippling" mode. Solid arrows indicate fluid velocity. (b) Perturbed fields and velocities—"tearing" mode. (c) Perturbed fields and velocities—gravitational mode.

characteristic times becoming infinite. The situation is quite analogous to the new modes which occur in hydrodynamics when the constraint of conservation of vorticity is removed by the presence of finite viscosity.

The growth rates of the "resistive" modes are sufficiently small on the hydromagnetic time scale so that the fluid motion is subsonic, i.e., incompressible. This feature is of critical importance in simplifying the analysis of the plane current layer: it permits us to consider the magnetic-field and velocity components within the ky plane independently of the components in the direction $\hat{n}$ normal to the ky plane. The reasons for this decoupling effect are readily seen.

The coordinate along $\hat{n}$ is ignorable; therefore, the field lines of the component $B_\perp$ in the $\hat{n}$ direction are not distorted during the perturbation. The only manner in which the magnitude of $B_\perp$ could affect the motion in the ky plane is by way of the magnetic pressure $B_\perp^2/8\pi$. The gradients of this pressure, however, merely tend to induce plasma compression or expansion. An incompressible fluid automatically provides compensating hydrostatic pressure gradients, so that there is no net effect on the dynamics. As for Ohm's law, there the resistive diffusion term does not couple the field components if the resistivity is isotropic, and the convective term couples the magnetic field and velocity components within the ky plane to each other. Finally, the two equations specifying $\mathbf{B}$ and $\mathbf{v}$ to be solenoidal hold as well for the vector components in the ky plane taken alone. Thus we have four equations for two unknown two-component vectors, and we may restrict ourselves in what follows to the analysis of the two-dimensional problem. Typical field and velocity components in the ky plane are illustrated in Fig. 2. We note parenthetically that the convenient reducibility of the three-dimensional finite-resistivity stability problem is wholly analogous to the reducibility of the finite-viscosity stability problem of ordinary hydrodynamics.[22]

[22] The similarity of the finite-resistivity and finite-viscosity problems was first pointed out to us by E. Reshotko. For a discussion of finite-viscosity instabilities, see C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, New York, 1955).

To understand the basic character of the unstable modes, let us consider the mechanism whereby the fluid resists detachment from flux lines. Starting with Ohm's law

$$\eta \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{63}$$

let us suppose that the fluid is moving but the flux lines are not, i.e.. $\mathbf{E} \equiv 0$. Then we find $\mathbf{j} = (\mathbf{v} \times \mathbf{B})/\eta$, with a resultant motor force

$$\mathbf{F}_s = \mathbf{j} \times \mathbf{B} = [\mathbf{B}(\mathbf{v} \cdot \mathbf{B}) - \mathbf{v}B^2]/\eta \tag{64}$$

that opposes the fluid motion. In the limit $\eta \rightarrow 0$, this force, of course, prevents any separate fluid motion from taking place. We note, however, that the restraining force becomes arbitrarily weak near the point where $\mathbf{B}$ vanishes, and this is the key to the situation. Since the quantity $\mathbf{B}$ in the present discussion refers only to the magnetic-field components in the ky plane, we can generally select $\mathbf{k}$ so that $\mathbf{B}$ has a null at any desired value of y. We may expect that detached fluid motion can take place within a region of order ϵa about such a null point. For each unstable mode, we will find a driving force $\mathbf{F}_d$ that dominates the restraining force $\mathbf{F}_s$ within the inner region, and that is itself dominated by $\mathbf{F}_s$ outside this region.

We can relate the "skin depth" ϵa to the growth rate of the instability. Since $\mathbf{F}_d$ is comparable in magnitude to $\mathbf{F}_s$, the rate at which work is done on the fluid is given by

$$P \sim -\mathbf{v} \cdot \mathbf{F}_s \sim v_y^2 (B')^2 (\epsilon a)^2/\eta, \tag{65}$$

where we have used $B \sim B'\epsilon a$. The driving force gives rise to motion both in the $\hat{y}$ and $\mathbf{k}$ directions, since $\nabla \cdot \mathbf{v} = 0$. In general the instability wavelength will be much larger than ϵa, and therefore the fluid kinetic energy in the $\mathbf{k}$ direction is dominant. Equating the rate of change of this energy to the driving power, we have

$$\omega\rho\, v_y^2/k^2(\epsilon a)^2 = v_y^2 (B')^2 (\epsilon a)^2/\eta.$$

The skin depth is then given by

$$\epsilon a \sim \left\{\frac{\omega\rho\eta}{k^2(B')^2}\right\}^{\frac{1}{4}} \tag{66}$$

which agrees with Eq. (34) when expressed in the appropriate dimensionless variables. To arrive at the instability growth rates, we must next determine ϵa by comparison of $\mathbf{F}_d$ with $\mathbf{F}_s$. For this purpose we turn to consideration of specific modes.

In the "rippling" mode of Fig. 2(a), the circulatory motion of the fluid creates a ridge of lower-resistivity fluid into which the local current is channeled. In other words, when a resistivity gradient exists, Ohm's law in its linearized form has an extra term

$$\eta_0 \mathbf{j}_1 = -\eta_1 \mathbf{j}_0 + \mathbf{v} \times \mathbf{B}, \tag{67}$$

where η_1 is given by the convective law

$$\eta_1 = -\mathbf{v} \cdot \nabla \eta_0 / \omega, \tag{68}$$

and where $\mathbf{E} \equiv 0$ has again been used, as is appropriate within the small region of decoupled flow. The η_1 term in Eq. (67) gives rise to a motor force

$$\begin{aligned} \mathbf{F}_{d_r} &= \mathbf{j}_1 \times \mathbf{B} \\ &= [(\mathbf{v} \cdot \nabla \eta_0)/\omega \eta_0] \mathbf{j}_0 \times \mathbf{B} \end{aligned} \tag{69}$$

that changes sign as $\mathbf{B}$ passes from one side of the null point to the other. Hence, $\mathbf{F}_{d_r}$ is a stabilizing force on the side of higher resistivity and a destabilizing force on the side of lower resistivity. An unstable mode is obtained if the region of decoupled flow lies on the lower-resistivity side and has a width ϵa such that the driving power

$$\mathbf{v} \cdot \mathbf{F}_{d_r} \sim v_y^2 \eta_0' (B')^2 (\epsilon a) / 4\pi \eta_0 \omega \tag{70}$$

just dominates $\mathbf{v} \cdot \mathbf{F}_s$ inside the region. Comparison of Eqs. (65) and (70) yields

$$\epsilon a \sim \eta_0' / 4\pi\omega. \tag{71}$$

From Eqs. (66) and (71) we can then obtain a growth rate that agrees with Eq. (50) and Table I. We note incidentally that the fluid flow and the perturbation current density are strongly peaked in the decoupled region, while the magnetic-field perturbation falls off over a region in y that is of order k^{-1}. In this outer region the fluid and field are well coupled, and a fluid motion of small magnitude accompanies the field perturbation.

We turn next to the gravitational interchange mode, which is quite similar in character to the "rippling" mode. In the presence of a mass-density gradient, and a y-directed gravitational field, the fluid motion gives rise to a force

$$\mathbf{F}_{d_g} = \rho_1 \mathbf{g} = (-v_y \rho_0' / \omega) \mathbf{g} \tag{72}$$

which is destabilizing if $\mathbf{g}$ points toward decreasing density. Comparison of $\mathbf{v} \cdot \mathbf{F}_{d_g}$ with Eq. (65) gives

$$\epsilon a \sim [\rho_0' g \eta / (B')^2 \omega]^{\frac{1}{4}}. \tag{73}$$

From Eqs. (66) and (73) we then obtain a growth rate that agrees with Eq. (59) and Table I. The mode that decouples the fluid and field most effectively in this case is the counter-circulatory mode shown in Fig. 2(c). In the infinite-conductivity case, such a fluid motion would lead to local compressions of $\mathbf{B}$, and so could not proceed unless $\mathbf{g}$ is large. This phenomenon is known as shear stabilization. In the mode of Fig. 2(c), the opposing flux components brought together at the null in $\mathbf{B}$ can cancel out, and so the mode can grow for arbitrarily small $\mathbf{g}$. As $\mathbf{g}$ increases, the region of substantial motion becomes wider, until conditions for infinite-conductivity instability are reached. We note that if $\mathbf{g}$ points in the stabilizing direction, the possibility exists of using $\mathbf{F}_{d_g}$ to overcome the driving force $\mathbf{F}_{d_r}$, thus stabilizing the "rippling" mode.

The "tearing" mode of Fig. 2(b) differs from the other two modes in that it is typically a long-wave rather than a short-wave mode relative to the dimension of the current layer. The driving force is due to the structure of the magnetic field outside the region of decoupled flow; i.e., the tendency of the sheet current to break up into a set of parallel pinches. [The nature of this force is readily perceived by applying the "rubber-band" argument to the diagram of Fig. 2(b), but as we are not dealing with a localized perturbation the argument is not quite so simple.] Even in the inner region, the flow is not perfectly decoupled, and a term

$$\mathbf{E} \sim (\omega B_y / k) \hat{n} \tag{74}$$

must be taken into account in Ohm's law. This term corresponds to the generation of the perturbation flux that links the field regions on either side of $\mathbf{B} = 0$. We have then

$$\eta_0 \mathbf{j}_1 = \mathbf{E}_1 + \mathbf{v} \times \mathbf{B} \tag{75}$$

and we must select ϵa so that the first term on the right in Eq. (75) dominates the second in the region of the partly decoupled flow. Using $\nabla \cdot \mathbf{B} = 0$, we have

$$\mathbf{j}_1 \sim (B_y'' / 4\pi k) \hat{n}. \tag{76}$$

For wavelengths that are much greater than the current-layer thickness a, we find

$$B_y'' \sim B_y' / \epsilon a \approx B_y / \epsilon k a \tag{77}$$

(cf. Eq. 28). If we now choose ϵa so that $\eta_0 \mathbf{j}_1 \sim \mathbf{E}_1$, we find

$$\epsilon a \sim \eta_0 / 4\pi k \omega. \tag{78}$$

The growth rate obtained from Eqs. (66) and (78) approximates the result of Eq. (57) and Table I. This analysis is applicable only for $ka \ll 1$, since otherwise Eq. (77) breaks down, B_y''/B_y becoming negative. Similarly, if B_y must vanish at a finite distance, Eq. (77) is altered, B_y''/B_y being diminished or made negative. The significance of these features in regard to stability is suggested also by Fig. 2(b): the closed lines of force cannot drive the instability by the "rubber-band" effect unless they are stretched out along the $\mathbf{B} = 0$ line, i.e., have small ka; and if conducting walls were introduced at finite values of y the lateral crowding of the lines of force would impede the driving mechanism.

It is of interest to note that the basic driving force for the "tearing" instability also exists in the infinite-conductivity equation (20). The displacement $\xi = \psi/F$ (and thus the instability itself) is precluded in the ordinary theory by the requirement that ξ be finite where $F = 0$.

VII. RELATION TO EXPERIMENT

Owing to the approximations made in Sec. II, the present results cannot be expected to provide a general basis for the prediction of instability phenomena in experimental plasmas. In particular, the use of the hydromagnetic approximation is not suitable for high-temperature plasmas, while for low-temperature plasmas the negelect of Ohmic heating, ionization effects, etc., becomes unjustifiable.

In spite of these shortcomings in rigor, the present analysis appears to be consistent with a wide range of experimental results and therefore permits speculation about the causes and remedies of observed instabilities. We discuss in this section a number of observed instability phenomena that are uncorrelated or inversely correlated with predictions of the infinite-conductivity hydromagnetic theory, but that can be accounted for at least qualitatively in terms of the present analysis. Further corrections and generalizations (some of which are discussed in the appendices) may provide a quantitative description of the stability behavior of current layers in experimental plasmas.

The "Tearing" Mode

Since the "tearing" mode is a long-wavelength instability that involves a considerable disturbance of the magnetic field of a current layer, it is particularly suitable for detailed experimental study. We will consider first the simple sheet pinch, which is characterized by $B_{z0} \equiv 0$.

In theta pinches where an initial B_z field is entrapped in plasma and compressed by a fast-rising B_z field of opposite sign,[23,24] a cylindrical current layer results that is fairly well represented by the plane-sheet-pinch model of the present analysis. Typical τ_R and τ_H values are 1–10 μsec and 0.01 μsec respectively, so that S is of order 100–1000. The "tearing" mode in this case would consist of a breakup of the cylindrical current layer into adjacent rings. The fastest growing wavelength [cf. Eq. (58)] is given by $\alpha \sim 0.2$, a value that is not sensitive to the exact magnitude of S. The corresponding growth rate [cf. Eq. (57)] is $p \sim 20$. Thus the predicted e-folding time is in the range 0.05–0.5 μsec.

In those experiments where the plasma volume is short in the z direction,[25] the current layer is found to collapse into a single ring, presumably because there is not adequate room and time for a full wavelength of the "tearing" mode to establish itself. A more satisfactory test of the theory is expected for theta pinches that are sufficiently long to accommodate a number of wavelengths at $\alpha \sim 0.2$. Under these conditions, recent experiments at Aldermaston[26] have demonstrated plasma breakup into as many as six rings, with an instability growth time of about 0.3 μsec. Even long reverse-field theta pinches are found stable under certain conditions,[26] which may be related to the effect of cylindrical geometry and external conductors (cf. Appendix G).

The gyro-orbits of particles in theta pinches are not very small compared with the dimension a of the current layer itself, let alone the dimension of discontinuity ϵa of the "tearing" mode. We note, however (cf. Sec. II, paragraph 1), that the "tearing" mode exists not only in the hydromagnetic limit but also in the collisionless limit, where the Vlasov equation is used directly.[18] It seems likely, therefore, that allowance for nonhydromagnetic effects is not crucial in the case of the "tearing" mode.

A second experimental embodiment of the simple sheet pinch is the Triax or tubular dynamic pinch.[27] In this case a reverse-B_θ layer is created. In the high-density and highly dynamic forms of this pinch (3 megamperes, 300 μD_2) that are usually employed, the "tearing" mode has not been seen, though an

[23] A. C. Kolb, C. B. Dobbie, and H. R. Griem, Phys. Rev. Letters 3, 5 (1959).
[24] H. A. B. Bodin, T. S. Green, G. B. F. Niblett, N. J. Peacock, J. M. P. Quinn, and J. A. Reynolds, Nuclear Fusion Suppl., Pt. 2, 521 (1962).
[25] V. Josephson, M. H. Dazey, and R. Wuerker, Phys. Rev. Letters 5, 416 (1960).
[26] H. A. B. Bodin (private communication, 1962).
[27] O. A. Anderson, W. A. Baker, J. Ise, Jr., W. B. Kunkel, R. V. Pyle, and J. M. Stone, in *Proceedings of the Second International Conference on Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 32, p. 150.

effort has been made to induce it.[28] This study is currently being extended. In a slower and weaker pinch (450 kiloamperes), the expected tearing along (axial) current-flow lines has been observed[29] at pressures of 30–300μ in deuterium and argon, with wavelengths and growth times that agree well with the present analysis.

The addition of a B_{x0} field to the simple reverse-B_{z0} sheet pinch evidently has no effect at all on the "tearing" mode if $k_x = 0$ (the "symmetric" case, where $F'' = 0$ at $F = 0$). Thus the "tearing" mode may occur in the Triax configuration even in the presence of an axial magnetic field. Evidence for such an instability has been found by magnetic probe measurements.[1]

In more general current layers, such as those of the "stabilized" and "inverse stabilized" pinches,[1] we can always choose our coordinates so as to transform the current layer into the basic model that is obtained for the Triax plus axial field. If we wish to look at the "symmetric case" with $k_x = 0$, we orient the x axis of the plane model along **B** at the midpoint of the current layer. If the thickness a of the current layer is small compared with its radius R, the outer solution of Sec. IV is readily adapted to cylindrical geometry (see Appendix I). In this manner we can show that a "stabilized pinch" with a sharp current layer is essentially always unstable against the "tearing" mode. A hard-core pinch with a large vacuum B_θ field and no null in the B_z field would have an advantage here, since in this limit k is forced to become large by the periodicity requirement along the axial coordinate, the limiting α for which the "tearing" mode can exist being reached when $B_\theta/B_z \sim R/a$.

A second advantage that is realizable in the hard-core pinch relates to the use of magnetic fields produced largely by external conductors. In extending the present results to nonplanar current layers, the quantities F', F'' require special interpretation. From the derivation of Eqs. (13) and (14), it is clear that F', F'' relate to the zero-order current in the plasma. In the planar case, a zero-order vacuum-field component has constant B_{x0}, B_{z0}, and thus cannot contribute to F', F''. In the nonplanar case, where a zero-order vacuum-field component may have nonzero derivatives of B_{x0}, B_{z0}, the contribution of the vacuum-field component to F', F'' must be specifically excluded. If the zero-order plasma current is small relative to currents in rigid conductors (e.g., currents in the central core and B_z-winding of a hard-core pinch), then the F', F'' terms in Eqs. (13), (14) tend to become small relative to the F terms. Accordingly, the "rippling" and "tearing" modes, which depend on the F' and F'' terms respectively, tend to be inhibited. The behavior of the gravitational interchange mode is given in Appendix I.

For the "stabilized pinch," an $m = 1$ mode conforming with the magnetic-field direction could be obtained even in the infinite-conductivity limit, so that for this configuration a detailed experimental study would be necessary to establish the occurrence of the resistive "tearing" mode. For an "inverse stabilized pinch" with $B_\theta \sim B_z$, an $m = 1$ mode conforming with the field has been found experimentally,[2] contrary to the prediction of the infinite-conductivity theory, and consistent with the present analysis. For $\tau_R \sim 20$ μsec, $\tau_H \sim 0.2$ μsec, the e-folding time of the "tearing" mode [cf. Eqs. (57), (58)] is about 2 μsec.

A striking feature of magnetic-probe traces taken on the "stabilized"[30] and "inverse stabilized"[2] pinch discharges is that magnetic turbulence is suppressed during the initial dynamic phase. The present analysis provides a possible explanation. In Sec. V we note that a sufficiently strong gravitational effect (i.e., an accelerational effect in the present case) will suppress the "tearing" mode in favor of the gravitational interchange mode. Especially in the presence of an oscillating gravitational field, only short-wave gravitational interchange modes tend to grow, with a resultant minimal disturbance of the magnetic field.

The Interchange Modes

In the limit of high S and small G, the "rippling" and gravitational modes grow preferentially at short wavelengths and with $\mathbf{k}\cdot\mathbf{B} = 0$, so that there is a minimal disturbance of the magnetic field. The main effects to be looked for experimentally are a fluctuating electric field transverse to **B** and a loss of hot plasma out of the current layer. The "rippling" mode interchanges high-conductivity against low-conductivity plasma, and the gravitational mode interchanges high-pressure against low-pressure plasma or permits decelerating plasma to pass across magnetic field.

Recent studies on Zeta[31] have shown that the dominant nonradiative energy loss takes place by

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[30] L. C. Burkhardt and R. H. Lovberg, in *Proceedings of the Second International Conference on Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 32, p. 29.
[31] W. M. Burton, E. P. Butt, H. C. Cole, A. Gibson, D. W. Mason, R. S. Pease, K. Whitman, and R. Wilson, IAEA Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, Austria, (1961), paper 60.

convection of hot plasma across magnetic field to the tube wall. The plasma convection is accompanied by fluctuating transverse electric fields with frequencies in the 10–100 kc range. The wavelengths are of order 20 cm in the direction across magnetic field, and are much greater in the direction along magnetic field. The parameters τ_R and τ_H are typically 3000 μsec and 1 μsec. For the "rippling" mode, we thus obtain $p \sim 100$ (cf. Eq. (50)). The resultant e-folding time of 30 μsec is perhaps somewhat too long to fit the data. The β value is of order 0.1, so that the G term and the $(\eta')^2$ term in Eq. (39) are of the same magnitude. The cooperation of the two driving mechanisms leads to a slightly enhanced growth rate. The stabilizing effect of heat conductivity on the "rippling" mode in Zeta (cf. Appendix F) begins to play an important role at electron temperatures above 10 eV.

The main difficulty in accounting for the Zeta results lies in the extremely short dimension of the region of discontinuity ($\epsilon a \sim 1$ cm) that is called for by the present analysis. The ion gyroradii in Zeta tend to be of this size or even larger. A nonhydromagnetic treatment of the region of discontinuity is therefore necessary to provide a rigorously valid model.

The attribution of the plasma loss in Zeta primarily to the "rippling" mode would have one especially engaging feature that deserves mention. Contrary to expectation from the ordinary theory of the interchange mode, the plasma in Zeta is *most* stable when the field lines in the central region of null shear come back on themselves on going once around the major circumference of the torus.[32] At these "magic number" points—the higher harmonics of the Kruskal limit—the periodicity condition around the major circumference permits an interchange mode to align itself perfectly with the null-shear magnetic field in the central region, which is advantageous for the growth of gravitational modes. We note, however, from Eq. (31) that the "rippling" mode is not perfectly aligned with the local magnetic field, but rather with the magnetic field at a point that is slightly displaced from the point of interchange. To generate the basic motor force of the instability, the perturbed current channel must make a small angle with respect to the field in the hot plasma. Thus the "magic number" regimes are generally unfavorable to the growth of the "rippling" mode.

A number of authors[15–17] have pointed out that the "rippling" mode is well suited to account for the pump-out[33] phenomenon in discharge tubes of the stellarator type. Using the present results for the growth rate [cf. Eq. (50)] and assuming typical parameters $\tau_R = 100$ μsec $\tau_H = 0.01$ μsec, we obtain $p \sim 100$. Due to the heat-conductivity effect (cf. Appendix F) the expected growth times of ~ 1 μsec become longer at electron temperatures above 10 eV. Again we note that the dimension ϵa of the discontinuity has an unrealistically small value: less than a millimeter. A nonhydromagnetic treatment is needed to give the true growth rate.

The "rippling" mode is not an inherent threat to the stellarator plasma-confinement scheme, since the discharge current can always be replaced, at least conceptually, by other heating methods. But even in the absence of current, the stellarator generally has some tendency toward gravitational interchanges, which is supposed to be suppressed by the shear of the magnetic field. For typical parameters of present experiments, the G term in Eq. (39) is much smaller than the η'^2 term but once the heating current is removed the pressure-driven resistive mode may become the dominant source of difficulty. The same remark applies to the stellaratorlike "Levitron"[1] (toroidal hard-core pinch). Hopefully, the nonhydromagnetic effects will serve to suppress the interchange mode in the limit of high conductivity. In particular, in reference 34 it is shown that, for a certain class of perturbations, a stabilizing charge-separation occurs due to finite Larmor radius. This leads to stability if $\omega_H/\omega_c < (kR_L)^2$ where ω_c is the cylotron frequency, and ω_H is the growth rate without correction for finite-Larmor-radius R_L. Since ω_H tends to be small for resistive instabilities, one might expect a strong stabilizing effect.

While the gravitational and resistivity-gradient effects are usually such as to collaborate in promoting instability in pinch and stellarator-type devices, the possibility exists of designing special regimes where the two effects are balanced against each other. For example, a stellarator discharge might be stabilized against the "rippling" mode if the absolute magnetic field strength were made to increase everywhere with radius. Such an effect can be achieved by giving the stabilizing windings an appropriate pitch. The required magnitude of stabilizing field is indicated by Eq. (60).

Direct observation of interchange modes has been

[32] E. P. Butt, Bull. Am. Phys. Soc. 7, 148 (1962).

[33] E. P. Goburnov, G. G. Dolgov-Savelev, K. B. Kartashev, V. S. Nukhovatov, V. S. Strelkov, and N. A. Yavlinski, IAEA Conference on Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, Austria, (1961), paper 223.

[34] M. N. Rosenbluth, N. A. Krall, and N. Rostoker, Nuclear Fusion Suppl., Pt. 1, 143 (1962).

possible in a linear "stabilized pinch" experiment.[35] Stereoscopic Kerr-cell photography through a screen electrode reveals luminous "streamers" that have various orientations during the dynamic and quasistatic phases of the pinch cycle. During the dynamic phase, the streamers are aligned with the magnetic field in the regions of highest g stress (i.e., nearly pure B_z field and nearly pure B_θ) field. During the quasistatic phase, helical streamers are seen, which are more nearly aligned with the mean magnetic field in the current layer, and which can be accounted for as a mixture of the "rippling" and "tearing" modes.

VIII. COMPUTATIONAL PROGRAM

To supplement the analytical treatment in the range of intermediate S and to obtain accurate results in general for specific choices of F and of the boundary conditions, an IBM 709 code is available. This code is based on Eqs. (2)–(6) in linearized form and makes use of Fourier analysis in space but not in time. Accordingly, the development of specific initial disturbances can be studied. The code is applicable to both unstable and overstable modes, and will be capable of incorporating Ohmic heating, ionization, and similar effects.

Preliminary results have been reported,[8] and a more exhaustive study is under way.

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APPENDIX A. EFFECT OF FLUID COMPRESSIBILITY

Allowance for fluid compressibility has two principal effects on the stability analysis. The equation of motion of the fluid is altered, and the first-order quantities η_1 and $(\mathbf{g}\rho)_1$ receive nonconvective contributions.

We begin by demonstrating that the dynamic effect of compressibility is generally negligible for the modes of Sec. V. The equation $\nabla\cdot\mathbf{v}_1 = 0$ is no longer valid now, and we replace W' by $W' + i\alpha\tau_R\nabla\cdot\mathbf{v}_1$ in Eq. (14) [Equation (13) is unaltered]. The extra term arises from the fact that $\nabla\cdot\mathbf{v}_1 \neq 0$ when one eliminates $(\mathbf{k}\cdot\mathbf{v}_1)$ from the equations of motion.

To find $\nabla\cdot\mathbf{v}_1$, we use the equation describing the pressure perturbation

$$\omega P_1 + \mathbf{v}_1\cdot\nabla P_0 + \gamma P_0\nabla\cdot\mathbf{v}_1 = [(\gamma-1)/(4\pi)^2] \cdot[\eta_1(\nabla\times\mathbf{B}_0)^2 + 2\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1)]. \qquad (A.1)$$

The effect of Ohmic heating has been included. Heat losses due to conduction and radiation have been neglected.

To determine the first-order pressure contribution P_1, we use the equation of motion in the form

$$\mathbf{k}\cdot\{\rho_0\omega\mathbf{v}_1 + \nabla P_1 - (4\pi)^{-1}[(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1 + (\nabla\times\mathbf{B}_1)\times\mathbf{B}_0] - (\rho\mathbf{g})_1\} = 0 \qquad (A.2)$$

which reduces to

$$P_1 + \frac{B_{\perp 0}B_{\perp 1}}{4\pi} + i\frac{B_{\parallel 1}}{4\pi}\frac{dB_{\parallel 0}}{k\,dy} - i\frac{\rho_0\omega}{k}v_\parallel = 0, \qquad (A.3)$$

where

$$B_{\perp 0} = \hat{y}\cdot(\mathbf{k}\times\mathbf{B}_0)/k = BH,$$
$$B_{\perp 1} = \hat{y}\cdot(\mathbf{k}\times\mathbf{B}_1)/k = (B/i\alpha)\,\chi,$$
$$B_{\parallel 0} = \mathbf{k}\cdot\mathbf{B}_0/k = BF,$$
$$B_{\parallel 1} = \mathbf{k}\cdot\mathbf{B}_1/k = (iB/\alpha)\psi'$$
$$v_\parallel = \mathbf{k}\cdot\mathbf{v}_1/k = -(1/k\alpha\tau_R)(W' + i\alpha\tau_R\nabla\cdot\mathbf{v}_1).$$

We will show below that in the "region of discontinuity" of Sec. V, which is the region of critical interest, the $B_{\parallel 1}$ and $v_\parallel$ terms are negligible. One finds also $|B_{\perp 0}B_{\perp 1}| \gg |B_{\parallel 0}B_{\parallel 1}|$.

Equation (A.3) then yields

$$P_1 = -B_{\perp 0}B_{\perp 1}/4\pi = -\mathbf{B}_0\cdot\mathbf{B}_1/4\pi \qquad (A.4)$$

as would be expected for a subsonic motion: the total (fluid plus magnetic) pressure remains approximately constant.

To evaluate the $B_{\perp 0}B_{\perp 1}$ term in Eq. (A.3), and to prove its predominance, we must find the perturbation-field amplitude χ. In the analysis of the incompressible case, it was unnecessary to obtain χ explicitly in order to find the dispersion relation. The solution was obtained in terms of ψ

[35] D. J. Albares and C. L. Oxley, Bull. Am. Phys. Soc. 7, 147 (1962).

and W, and the quantity χ could then be derived, if desired, by means of Eqs. (8) and (9). Since we will show that compressibility has a negligible effect, we may proceed in precisely the same manner, first obtaining χ for the incompressible case, and then using the result in Eqs. (A.3) and (A.1) to verify the effect of compressibility. The terms involving $\nabla\cdot\mathbf{v} \neq 0$ may be easily checked subsequently to be small.

From Eqs. (8) and (9) we obtain

$$(\tilde{\eta}\chi')' = \chi[\alpha^2\tilde{\eta} + p + (\alpha^2S^2/\tilde{\rho}p)F^2] - WH' - (1/p)[(W\tilde{\eta}'H') - \alpha^2W\tilde{\eta}'H] + (\alpha^2S^2/\tilde{\rho}p)FH'\psi, \quad \text{(A.5)}$$

an equation that is similar in structure to Eqs. (13) and (14). In the "outer region" of Sec. IV, we have

$$\chi F = -H'\psi \quad \text{(A.6)}$$

which may be used with Eqs. (19), (A.3), and (A.1) to demonstrate strictly incompressible flow, as might be expected. In the region of discontinuity, the terms of Eq. (A.5) involving $\tilde{\eta}'$ are of order $p^{\frac{1}{2}}S^{-\frac{1}{2}}$ relative to the ψ term, and may be neglected. Similarly, the term WH' is of order $p^{\frac{1}{2}}S^{-1}$ relative to the ψ term and may be neglected except in the case $G \sim 1$, in which case the two terms are comparable. Transforming to the variable $\theta_1 = (\mu - \mu_0)/\epsilon$, and making the usual approximations in the region of discontinuity (cf. Sec. V), we obtain

$$d^2\chi/d\theta_1^2 - \tfrac{1}{4}\theta_1^2\chi = (H'/4\epsilon F')\psi\theta_1. \quad \text{(A.7)}$$

Thus χ behaves much like W. From Eq. (A.7) we infer

$$\chi = O[(H'/\epsilon F')\psi] \quad \text{(A.8)}$$

in the region of discontinuity.

We may now return to evaluate the magnitude of terms in Eq. (A.3). From Eq. (A.8), we see directly that the $B_{\nu 1}$ term is negligible. From Eqs. (A.8) and (35), it follows that the $v_{\parallel}$ term is of order $p^{7/4}S^{-\frac{1}{2}}$, and is therefore negligible. We have now proved Eq. (A.4). Using Eqs (A.4), (A.8), and the zero-order relation $P_0' \approx -B^2HH'/4\pi$, we may write

$$P_1 = iO(\psi P_0'/\alpha\epsilon F'). \quad \text{(A.9)}$$

We may now estimate the magnitude of the terms of Eq. (A.1) in the region of discontinuity. The η_1 term is negligible relative to the ∇P_0 term, since

$$\eta_1(\nabla\times\mathbf{B}_0)^2 = O(\mathbf{v}_1\cdot\nabla P_0/p)$$

where we have used Eq. (11). For the η_0 term we find

$$\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1) = O(\omega P_1/p\epsilon)$$

so that it is of the same magnitude as the P_1 term for the "rippling" mode [cf. Eq. (53)], but is negligible for the low-α "tearing" mode [cf. Eqs. (36) and (56)] and the gravitational interchange mode. Next we compare the P_1 and ∇P_0 terms. From Eq. (A.9) we have

$$\omega P_1 = iO(p\psi P_0'/\epsilon\alpha F'\tau_R) \quad \text{(A.10)}$$

and we see directly that

$$\mathbf{v}_1\cdot\nabla P_0 = iO(WP_0'/\alpha\tau_R). \quad \text{(A.11)}$$

Using Eq. (35) and remembering that $U = 0(\psi)$, we conclude that the P_1 and ∇P_0 terms are of the same order.

Finally we may evaluate the correction to W' in Eq. (14) that results from compressibility. As we have noted in connection with Eq. (A.6), this correction is insignificant in the outer region. In the region of discontinuity, we have from Eq. (A.1), and from the remarks on the relative magnitude of its terms

$$|i\alpha\tau_R\nabla\cdot\mathbf{v}_1| = O[(P_0'/\gamma P_0)W] = O[(\epsilon P_0'/\gamma P_0)W'] \ll |W'|. \quad \text{(A.12)}$$

Thus, allowance for fluid compressibility does not directly affect the fluid motion involved in the modes of Sec. V.

The "tearing" mode is thus completely unaffected. For the "rippling" and gravitational interchange modes, which depend on the nature of Eqs. (11) and (12), there will generally be indirect compressional effects, since we see from Eq. (A.12) that the compressional changes in ρ and η are comparable to the purely convective changes. Accordingly, we write the equations

$$\omega\rho_1 + \mathbf{v}_1\cdot\nabla\rho_0 = -\rho_0\nabla\cdot\mathbf{v}_1, \quad \text{(A.13)}$$

$$\omega\eta_1 + \mathbf{v}_1\cdot\nabla\eta_0 = \tfrac{3}{2}(\gamma - 1)\eta_0 \cdot\left[\nabla\cdot\mathbf{v}_1 - \frac{\eta_1(\nabla\times\mathbf{B}_0)^2 + 2\eta_0(\nabla\times\mathbf{B}_0)\cdot(\nabla\times\mathbf{B}_1)}{(4\pi)^2P_0}\right], \quad \text{(A.14)}$$

where we have used the plasma properties $\eta \sim T^{-\frac{3}{2}}$ and $\rho T \sim P$.

We begin with the standard gravitational-interchange case, i.e., $\tilde{\eta}' = F'' = 0$, and $|G/(F')^2| \ll 1$. From Eq. (A.1) (neglecting the Ohmic-heating terms) and Eqs. (A.4) and (A.13), we obtain as the modified version of Eq. (33)

$$d^2U/d\theta^2 + U\{\Lambda(1 - L_s) - \tfrac{1}{4}\theta^2\} = \psi\theta - (4\epsilon F'\Lambda L_s/H')\chi \quad \text{(A.15)}$$

where

$$L_g = \rho_0 P_0'/\gamma \rho_0' P_0.$$

Equations (32), (A.7), and (A.15) can readily be solved by the method of Sec. V, and yield the eigenvalue equation

$$\Delta' = \frac{\pi 2^{7/2}\Omega}{1 - L_g} \cdot \left[\frac{\Gamma\{\frac{3}{4} - \frac{1}{2}\Lambda(1 - L_g)\}}{\Gamma\{\frac{1}{4} - \frac{1}{2}\Lambda(1 - L)_g\}} - L_g \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]. \quad \text{(A.16)}$$

We note that Λ is not much altered for $L_g < 1$; in the case $L_g > 1$, however, we now find unstable modes for both positive and negative G. (As will emerge in Appendix I, the dependence on compressibility is a peculiarity of the "true" gravitational mode, which disappears when G is to be interpreted as arising from a pressure gradient and curved field.)

For the "rippling" mode we proceed in a similar manner, now setting $G = 0$ and including Ohmic-heating effects. We obtain

$$\frac{d^2U}{d\theta_c^2} + U\{\Lambda(1 - L_r)^2 - \tfrac{1}{4}\theta_c^2\} = \psi(\theta_c - \delta_c) - \frac{4\epsilon F'\Lambda L_2}{H'}\chi - 3\tilde{\eta}^2\frac{(\gamma - 1)H'}{\gamma\beta_0\epsilon p^2}\frac{d\chi}{d\theta_c}(\theta_c - \delta_{1c}) \quad \text{(A.17)}$$

where

$$\beta_0 = 4\pi P_0/B^2,$$

$$\theta_c = \frac{1}{\epsilon}\left[\mu - \mu_0 + \frac{\tilde{\eta}'}{2p}(1 - L_r)\right],$$

$$\delta_c = \frac{1}{4\Omega}\left\{\frac{F''}{F'} + \frac{\tilde{\eta}'}{2\tilde{\eta}}(1 - L_r)\right\},$$

$$\delta_{1c} = \frac{1}{8\Omega}\frac{\tilde{\eta}'}{\eta}(1 - L_r),$$

$$L_r = -\frac{3(\gamma - 1)\eta_0 P_0'}{2\gamma\eta_0' P_0}.$$

The equations for the "rippling" mode are more difficult to solve formally, because the homogeneous part of Eq. (A.17) involves θ_c, while that of Eq. (A.7) involves θ_1. The main features of the result are evident, however. Since $\epsilon \sim 1/p$ for the "rippling" mode, we see that the χ and $d\chi/d\theta$ terms in Eq. (A.17) are of the same order in S and α as the ψ term. For $\alpha \gg 1$, we know from Sec. V that the U term becomes of order α relative to the ψ term, because $(n + \frac{1}{2} - \Lambda) \to 1/\alpha$ in Eq. (43). Thus, for the case of maximum interest, where S and α are large, we have simply

$$\Lambda = (m + \tfrac{1}{2})/(1 - L_r)^2, \quad m = 1, 2, 3 \cdots. \quad \text{(A.18)}$$

Since generally $P_0'\eta_0' < 0$, we will have $L_r > 0$, so that the growth-rate is reduced. For $L_r > 1$, the pressure-gradient effect dominates the resistivity-gradient effect in Eq. (A.14), and we obtain a new "rippling" mode that resembles the old one in every respect, except that η_0'/η_0 is to be replaced by $\frac{3}{2}[(\gamma - 1)/\gamma]P_0'/P_0$. Note that, in the case of maximum interest considered here, the Ohmic-heating effects do not play a role.

The analysis in this appendix has been carried out for H, $H' \neq 0$ at $F = 0$. If either H or H' has a null at $F = 0$, then it is easy to see that the compressibility effects become completely negligible as $S \to \infty$. If H' has a null, the Ohmic-heating effects become completely negligible as $S \to \infty$. (These remarks, of course, cover the important special case of unsheared field, where $H \equiv 0$.)

APPENDIX B. LOW-CONDUCTIVITY LIMIT

For $S \ll 1$, unstable modes cannot grow faster than ordinary resistive diffusion. In order that a zero-order equilibrium may exist, we therefore require that Eq. (15) be satisfied. For convenience we let $\tilde{\eta}F' = 1$. We will treat the case $G \equiv 0$.

To make a general estimate of p, we note from Eq. (18) that W^2 must be of order $S^2\psi^2$ or less. If we had $p \gg S$, Eq. (13) would reduce to

$$\psi'' - (\alpha^2 + pF')\psi = 0 \quad \text{(B.1)}$$

from which follows

$$\int_{\mu_1}^{\mu_2} d\mu\, [(\psi')^2 + (\alpha^2 + pF')\psi^2] = 0 \quad \text{(B.2)}$$

so that $p < 0$. Unstable modes are therefore characterized by $p \leq 0(S)$, which means that they grow on the hydromagnetic rather than on the resistive time scale.

The case $S < 1$ may be applied to liquid-metal experiments and to some experiments with dense, low-temperature plasmas of heavy ions. The former application has been investigated exhaustively by Murty,[9] who includes surface-tension and gravitational forces, and uses the slab model

$$F' = 1, \quad |\mu| < 1; \qquad F' = 0, \quad |\mu| > 1, \quad \text{(B.3)}$$

together with the specification $\tilde{\rho} = F'$.

We will begin with a more general treatment. As $S \to 0$, $p \to 0$, Eqs. (13) and (16) reduce to

$$\psi'' - \alpha^2\psi + vF'' = 0, \quad \text{(B.4)}$$

$$(\tilde{\rho}v')' - \alpha^2\tilde{\rho}v + (\alpha^2 S^2/p^2)F''(vF + \psi) = 0, \quad \text{(B.5)}$$

where $W = pv$. There are two characteristic cases: $\alpha^2 \gg |F''/F|$, for which the ψ term in Eq. (B.5) is negligible; and $\alpha^2 \ll |F''/F|_{\max}$, which includes Murty's model.

For $\alpha^2 \gg |F''/F|$, we will set $\bar{\rho} = 1$, and Eq. (B.5) then becomes

$$v'' + \alpha^2 v[-1 + (S^2/p^2)FF''] = 0. \qquad (B.6)$$

We note that short-wave perturbations are now localized near the point μ_r, where $(FF'')' = 0$ rather than near $F = 0$; and we expand

$$FF'' = f_0 + f_2\mu_2^2$$

where $\mu_2 = \mu - \mu_r$. Equation (B.6) then has the same form as Eq. (41). The fastest growing mode is given by

$$v = \exp[-\tfrac{1}{2}\alpha(-f_2/f_0)^{\frac{1}{2}}\mu_2^2]$$

and the corresponding eigenvalue relation is

$$p = Sf_0^{\frac{1}{2}}. \qquad (B.7)$$

The condition for instability is $f_0 > 0 > f_2$. To illustrate what this condition means, let us consider the model

$$F = F_0 + \tanh \mu_0$$

where F_0 is an arbitrary constant. Then we have

$$FF'' = -2(\tanh \mu/\cosh^2 \mu)(F_0 + \tanh \mu).$$

For $F_0 = 0$, we find $FF'' < 0$; therefore there are no modes of the symmetric "tearing" type. For $F_0^2 \ll 1$, we find $\mu_r = -\frac{1}{2}F_0$, $f_0 = \frac{1}{2}F_0^2$, $f_2 = -2$. For $F_0^2 \gg 1$, we find $\mu_r = \mp\sinh^{-1}(2^{-\frac{1}{2}})$, with the sign of μ_r to be taken opposite to that of F_0 for instability, in which case $f_0 = 2\,|F_0|\,3^{-\frac{1}{2}}$, $f_2 = 8\,|F_0|\,3^{-\frac{1}{2}}$. Thus the "rippling" mode exists for $F_0 \neq 0$, and grows most rapidly for $F_0^2 \gg 1$.

We turn now to the second characteristic case, $\alpha^2 \ll |F''/F|_{\max}$. The model of Eq. (B.3) is typical of this case, and will be adopted here. For simplicity, we will specify $\bar{\rho} = 1$ everywhere. This density profile is somewhat more suitable for the plasma application than Murty's, and is well suited to describe liquid-metal layers suspended in a density-matching oil.[36] Except at the two points where $F'' \neq 0$, the solution to Eqs. (B.4) and (B.5) have the form ψ, $v \sim e^{\pm\alpha\mu}$, so that the problem is a purely algebraic one. To obtain the dispersion relation, it is convenient to write Eq. (B.3) in the form

$$F'' = -\delta(\mu - 1) + \delta(\mu + 1)$$

$$F = F_1 = F_0 - 1, \qquad \mu < -1;$$

$$F = F_0 + \mu, \qquad |\mu| < 1;$$

$$F = F_2 = F_0 + 1, \qquad \mu > 1.$$

The resultant equations involve ψ_1, ψ_2, v_1, and v_2, where the subscripts refer to values of ψ and v at $\mu = -1$ and $+1$ respectively. Eliminating ψ_1, ψ_2, one obtains the two-dimensional homogeneous vector equation

$$v_2[2p^2/\alpha S^2 - (1/2\alpha)(1 - e^{-4\alpha}) + F_2] - v_1F_1e^{-2\alpha} = 0, \qquad (B.8)$$

$$v_1[2p^2/\alpha S^2 - (1/2\alpha)(1 - e^{-4\alpha}) - F_1] + v_2F_2e^{-2\alpha} = 0, \qquad (B.9)$$

so that

$$p^2/S^2 = \tfrac{1}{4}(1 - e^{-4\alpha}) - \tfrac{1}{2}\alpha\{1 \pm [1 + (F_0^2 - 1)(1 - e^{-4\alpha})]^{\frac{1}{2}}\}. \qquad (B.10)$$

The eigenmodes are described by

$$v_2 = v_1\,\frac{(F_0 - 1)e^{-2\alpha}}{F_0 \mp [1 + (F_0^2 - 1)(1 - e^{-4\alpha})]^{\frac{1}{2}}}. \qquad (B.11)$$

The typical "tearing" mode is found for $F_0 = 0$, where Eq. (B.11) yields $v_2 = \pm v_1$, and may be identified with the antisymmetric-v eigenmode. The typical "rippling" mode is found for $F_0^2 \gg 1$, so that $v_2 = v_1e^{-2\alpha}[1 \mp (1 - e^{-4\alpha})^{\frac{1}{2}}]^{-1}$, and may be identified with the eigenmode involving the positive square root, when $F_0 > 0$.

For $\alpha \ll 1$, the eigenvalues are

$$p^2/S^2 = 0, \alpha, \qquad (B.12)$$

and the eigenmodes are characterized by

$$v_2/v_1 = 1, \quad (F_0 - 1)/(F_0 + 1). \qquad (B.13)$$

The first of these modes, describing a simple displacement of the current layer, is the low-α limit of the "rippling" mode; the second covers the "tearing" mode.

For $\alpha \gg 1$, we have

$$p^2/S^2 = \tfrac{1}{4} - \tfrac{1}{2}\alpha(1 \pm F_0). \qquad (B.14)$$

In the special case $F_0 = 0$, Eq. (B.14) gives a single solution

$$p^2/S^2 = \tfrac{1}{4} - \tfrac{1}{2}\alpha \qquad (B.15)$$

so that both the antisymmetric-v "tearing" mode and the symmetric-v mode are stable. Equation (B.14) indicates that instability can be obtained only for $F_0^2 > 1$, and that the growth rate increases with F_0^2. For $F_0^2 \to \infty$, we have

[36] S. A. Colgate, H. P. Furth, and F. O. Halliday, Revs. Mod. Phys. 32, 744 (1960).

$$p^2/S^2 = \mp\tfrac{1}{2}\alpha F_0, \tag{B.16}$$

$$v_2/v_1 = 2e^{2\alpha}, \qquad \tfrac{1}{2}e^{-2\alpha}. \tag{B.17}$$

For positive F_0, the second of these eigenmodes is unstable and corresponds to the "rippling" mode.

The results we have obtained for the $\alpha^2 \ll |F''/F|_{\max}$ case are substantially similar to those of Murty.[9] We note that, for the "tearing" mode, p has a maximum of order S. For the "rippling" mode, Eq. (B.16) indicates that p is of order $\alpha^{1/2}S$, in contrast with Eq. (B.7), where p is found to be of order S. Thus the increase in growth rate at short wavelengths predicted by Eq. (B.16) is seen to be dependent on the discontinuities of F' assumed in the zero-order configuration. The long-wave "tearing" mode is not dependent on the exact structure of F, and similar growth rates would be expected for continuous-F' models.

APPENDIX C. THE LIMIT $\alpha^2\epsilon^2 > 1$

We consider here the case where the instability wavelength is smaller than the region of discontinuity. This limit is more of mathematical than of physical interest, since ordinary resistive diffusion proceeds more rapidly at such small wavelengths than does the instability. For $\alpha^2\epsilon^2 > 1$, it is convenient to write Eq. (32) in the form

$$\psi = \frac{-\Omega}{2\alpha}\int_{-\infty}^{\infty} d\theta_1\, e^{-\epsilon\alpha|\theta-\theta_1|}\, U(\theta_1 + \delta_1)$$

$$\approx \frac{-\Omega}{\epsilon\alpha^2}\, U(\theta + \delta_1) \tag{C.1}$$

valid if $\Omega/\epsilon\alpha^2 \ll 1$. Thus we see that ψ is no longer constant in the region of discontinuity, but varies as strongly as U.

The ψ term in Eq. (33) becomes negligible, and the eigenvalues Λ lie very close to $\frac{1}{2}, \frac{3}{2}, \frac{5}{2} \cdots$.

For the "rippling" mode we now have from Eq. (39), when $\alpha^2\epsilon^2 \gg 1$,

$$p = (S\bar{\eta}'\ |F'|/2\bar{\eta}^{1/2}\bar{\rho}^{1/2})^{2/3}. \tag{C.2}$$

For the gravitational interchange mode, we may drop the restriction Eq. (45) on the magnitude of G, and obtain from Eq. (39), when $\alpha^2\epsilon^2 \gg \Lambda$,

$$p = SG^{1/2}/\bar{\rho}^{1/2}. \tag{C.3}$$

Using Eqs. (C.2) and (C.3), we may verify the validity of the initial assumption that $\Omega/\epsilon\alpha^2 \ll 1$ if $\alpha^2\epsilon^2 > 1$.

APPENDIX D. LOW-α LIMIT FOR THE "TEARING" MODE

Neglecting terms of order α^2 in Eqs (13) and (14), we will analyze the "tearing" mode without invoking the constant-ψ approximation which was used in Sec. V, and which is inapplicable for $\alpha \to 0$. We will neglect G and $\bar{\eta}'$, and restrict ourselves to the symmetric "tearing" mode, where $F'' = 0$ at $F = 0$.

In the region of discontinuity, we will set $\bar{\eta} = \bar{\rho} = F' = 1$ for convenience, so that $F = \mu$. Eqs. (13) and (14) then yield

$$z''' = pz' + (\alpha^2S^2/p)(\mu^2z' + 4\mu z) \tag{D.1}$$

where

$$z = \psi'' = p\psi + WF'\mu. \tag{D.2}$$

Introducing a new independent variable

$$\theta_1 = \mu(\alpha^2S^2/p)^{1/4}$$

and defining

$$\lambda = p^{3/2}/\alpha S$$

we obtain

$$d^3z/d\theta_1^3 = (\lambda + \theta_1^2)\, dz/d\theta_1 + 4\theta_1 z, \tag{D.3}$$

to be solved subject to the boundary conditions

$$\left.\begin{aligned} z &= 1,\\ z' &= 0,\end{aligned}\right\}\ \theta_1 = 0,$$

$$z = 0, \qquad \theta_1 = \infty.$$

We observe that there are two well-behaved solutions at $\theta_1 = \infty$, namely $z \sim \theta_1^{-4}$ and $z \sim \exp(-\tfrac{1}{2}\theta_1^2)$, so that Eq. (D.2) can always be solved. The relationship between the solution z and the quantity Δ' of Eq. (21) is given by

$$\Delta' = 2\lim_{\theta_1\to\infty}\left(\frac{\psi'}{\psi - \mu\psi'}\right) \tag{D.4}$$

where the denominator represents the intersection of the asymptote of ψ with the ψ axis at $\mu = 0$. Using Eq. (D.2), we have

$$\Delta' = 2p\int_0^\infty z\,d\mu \bigg/ \left(1 - p\int_0^\infty \mu z\,d\mu\right). \tag{D.5}$$

If we let $\Delta' = 2/\alpha$ (cf. Eq. 28) and define $p_1 = S^{-1/2}p$, $\alpha_1 = S^{1/4}\alpha$, we obtain from Eq. (D.5) the eigenvalue relation

$$p_1^{7/4}H(p_1/\alpha_1) = 1 \tag{D.6}$$

where

$$H(\lambda) = \lambda^{-1/2}\int_0^\infty z\,d\theta_1 \bigg/ \left(1 - \lambda\int_0^\infty \theta_1 z\,d\theta_1\right) \tag{D.7}$$

with the integrals to be determined by means of the solution of Eq. (D.3). We note that near $\lambda = 0$, $H \approx \lambda^{-1/2}$, so that $p_1 \approx \alpha_1^{-2/5}$, or $p \approx (S/\alpha)^{2/5}$, the

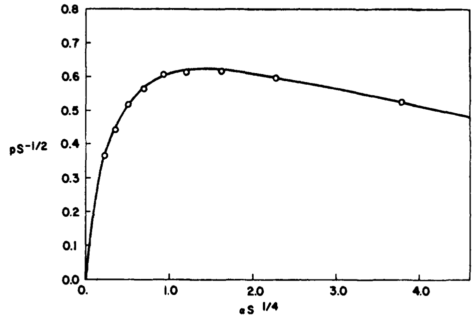

FIG. 3. Growth rates for long wavelength "tearing" mode.

result obtained in Sec. V. It is also easy to verify by substitution that for $\lambda = 1$, we have $z = \exp(-\frac{1}{2}\theta_1^2)$ and $H(\lambda) = \infty$. Thus p_1 is zero both for $\lambda = 0$ and $\lambda = 1$, and has a maximum value of order unity somewhere in between. This result verifies the conjecture made in Sec. V that the maximum of p with respect to α is of order $S^{\frac{1}{2}}$. Eq. (D.4) has been integrated numerically on the UCSD-CDC 1604. The solution was used to construct Fig. 3, which describes the detailed behavior of p in the long wavelength limit.

APPENDIX E. GENERAL ANALYSIS OF GRAVITATIONAL INSTABILITIES

Basic Equations

In Sec. V we have treated the gravitational interchange instability for the case where $G(F')^{-2}$ is sufficiently small so that ψ is constant in a region R_0 of width $\epsilon_0 \sim (1 + |\Delta'|)^{-1}$ about the point μ_0, where $F = 0$. The object of this appendix is to derive conditions on $G(F')^{-2}$ for which the constant-ψ approximation is justified, and to extend the analysis to the case of stronger gravitational fields.

As in Eqs. (32) and (33), we will treat a region about the point μ_0 in which F', $\tilde{\eta}$, G, and $\tilde{\rho}$ are constant, while F is approximated by $F'\mu_1$, where $\mu_1 = \mu - \mu_0$. For simplicity we will treat the pure gravitational-instability case where F'', $\tilde{\eta}' = 0$, so that the "tearing" and "rippling" modes are absent.

The analysis is valid in the range $\mu_1^2 \ll 1$. Thus it applies to high-G modes of short wavelength, i.e., $\alpha > 1$, and it also permits us to assess the constancy of ψ near μ_0 for low-G modes of arbitrary wavelength. High-G modes of long wavelength are not covered, but these are of lesser practical interest.

Eqs. (13) and (14) now reduce to the form

$$\psi''/\alpha^2 = \psi(1 + p/\alpha^2) + (W/\alpha^2)F'\mu_1, \qquad (E.1)$$

$$W''/\alpha^2 = W[1 - S^2G/p^2 + (F')^2\mu_1^2S^2/p] + \psi S^2F'\mu_1, \qquad (E.2)$$

where we have set $\tilde{\eta} = \tilde{\rho} = 1$.

We make the Fourier transform

$$\psi = \int_{-\infty}^{\infty} dk_0\, \psi_k e^{ik_0\mu_1},$$

$$W = \int_{-\infty}^{\infty} dk_0\, W_k e^{ik_0\mu_1},$$

and obtain

$$\psi_k\left(\frac{k_0^2}{\alpha^2} + 1 + \frac{p}{\alpha^2}\right) + i\frac{F'}{\alpha^2}\frac{dW_k}{dk_0} = 0, \qquad (E.3)$$

$$W_k\left(\frac{k_0^2}{\alpha^2} + 1 - \frac{S^2G}{p^2}\right) - \frac{(F')^2S^2}{p}\frac{d^2W_k}{dk_0^2} + iF'S^2\frac{d\psi_k}{dk_0} = 0. \qquad (E.4)$$

Eliminating ψ_k, we have

$$\frac{(F')^2S^2}{p}\frac{d}{dk_0}\left(\frac{k_0^2 + \alpha^2}{k_0^2 + \alpha^2 + p}\frac{dW_k}{dk_0}\right) - W_k\left(\frac{k_0^2}{\alpha^2} + 1 - \frac{S^2G}{p^2}\right) = 0. \qquad (E.5)$$

In our usual limit $S \to \infty$, where $p \gg 1$, α^2, we may reduce Eq. (E.5) to standard form

$$\frac{d}{dk_1}\left\{\frac{k_1^2 + \sigma}{1 + k_1^2}\frac{dW_k}{dk_1}\right\} - (Ak_1^2 - D)W_k = 0, \qquad (E.6)$$

where

$$k_0 = p^{\frac{1}{2}}k_1, \qquad \sigma = \alpha^2/p \ll 1,$$

$$A = p^3/(F')^2S^2\alpha^2, \qquad D = G/(F')^2 - A\sigma.$$

We will confine ourselves to the case $G > 0$. Since $\sigma \ll 1$, we may split the analysis of Eq. (E.5) into two overlapping ranges: $k_1^2 < 1$ and $k_1^2 > \sigma$.

For $k_1^2 < 1$, we let $k_2 = \sigma^{-\frac{1}{2}}k_1 = \alpha^{-1}k_0$, and obtain

$$\frac{d}{dk_2}\left[(1 + k_2^2)\frac{dW_k}{dk_2}\right] - (A\sigma k_2^2 - D)W_k = 0. \qquad (E.7)$$

Low-G Case

At this point we turn specifically to consideration of the case where $G(F')^{-2} < \frac{1}{4}$. As we will see, the approximation $A\sigma \ll 1$ is appropriate to this case. Eq. (E.7) then reduces to the Legendre equation. The general solution is

$$W_k = C_1P_h(ik_2) + C_2P_h(-ik_2) \qquad (E.8)$$

where

$$h = \tfrac{1}{2}[-1 + (1 - 4D)^{\frac{1}{2}}] < 0. \qquad (E.9)$$

We note that the region $k_2 < 1$ is related to the "outer" region of Sec. IV, and that the choice of C_1 and C_2 will reflect the outside boundary conditions, e.g., for a symmetric layer we would have $C_1 = -C_2$. For large k_2 the asymptotic form of the Legendre function gives

$$W_t \approx L_1 k_2^h + L_2 k_2^{-(h+1)} \qquad \text{(E.10)}$$

where the constants $L_{1,2}$ are determined from the C's. For $D < \frac{1}{4}$, W_t decays at large k_2 (with the first term predominating), while for $D > \frac{1}{4}$, W_t is oscillatory. Since the oscillatory behavior is not acceptable, it follows that the $A\sigma \ll 1$ approximation is consistent only with the case $D < \frac{1}{4}$, [i.e., the case $G(F')^{-2} < \frac{1}{4}$].

Continuing with the analysis of the $D < \frac{1}{4}$ case, we proceed to solve Eq. (E.6) in the range $k_1^2 > \sigma$, using $W_t \approx k_1^h$ to give the behavior of W_t for $\sigma < k_1^2 \ll 1$. The appropriate form of Eq. (E.6) is

$$\frac{d}{dk_1}\left\{\frac{k_1^2}{1+k_1^2}\frac{dW_t}{dk_1}\right\} - (Ak_1^2 - D)W_t = 0 \qquad \text{(E.11)}$$

and a solution is

$$W_t = k_1^h e^{(\frac{1}{2}h)k_1^2} \qquad \text{(E.12)}$$

with eigenvalue

$$A = h^2, \qquad \text{(E.13)}$$

clearly the lowest eigenvalue. Hence, for the case $G(F')^{-2} < \frac{1}{4}$ we have

$$p = \left(S\alpha F'\left\{\frac{1 - [1 - 4G(F')^{-2}]^{\frac{1}{2}}}{2}\right\}\right)^{\frac{2}{3}}. \qquad \text{(E.14)}$$

In the low-G limit, Eq. (E.14) reduces to Eq. (59) for $\Lambda = \frac{1}{2}$, thus verifying our use of the constant-ψ approximation. However, this verification of the results of Sec. V is limited to the case $\alpha \gg 1$, since only in this limit is the instability localized, so that we may neglect F'', set $F = F'\mu$, etc.

For $\alpha < 1$, ψ increases away from μ_0 [cf. Eq. (26)], and the G-term in Eq. (20) may become important. In particular, if G is constant, we must require

$$G/(F_{\pm\infty})^2 \ll \alpha^2 \qquad \text{(E.15)}$$

in order to be able to neglect the contribution of the G-term at large μ.

High-G Case

To study the case $G(F')^{-2} > \frac{1}{4}$, i.e., where the Suydam criterion is violated, we return to the range $k_1^2 < 1$ and to Eq. (E.7) with $A\sigma$ finite. We note that the finiteness of $A\sigma$ implies $p \sim S$, as we would expect for an instability that exists in the infinite-conductivity limit. Equation (E.16) has been integrated numerically, and the results are given in Fig. 4. The main features of the solution may be deduced analytically.

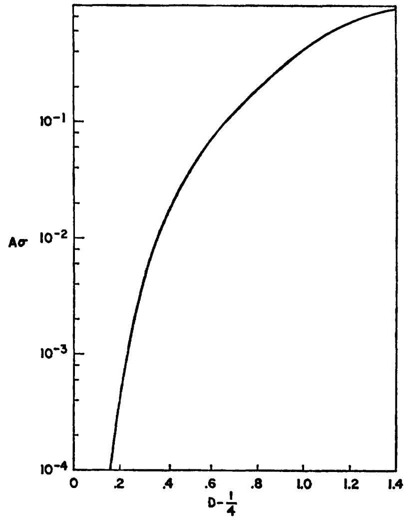

FIG. 4. Plot of eigenvalues of Eq. (E.16) for growth rates of hydromagnetically unstable gravitational mode.

For $G(F')^{-2} > \frac{1}{4}$ the complete solution W_t is obtained within the range $k_1^2 < 1$. (The range $k_1^2 > \sigma$ corresponds to the "region of discontinuity" of Section V, and disappears for infinite-conductivity modes.) Thus we have the boundary condition that the solution of Eq. (E.7) should vanish at $\pm\infty$. If we define $k_2 = \sinh z$, then Eq. (E.7) can be reduced to the form

$$d^2W_t/dz^2 + [(D - \tfrac{1}{4}) - \tfrac{1}{4}\operatorname{sech}^2 z - A\sigma \sinh^2 z]W_t = 0. \qquad \text{(E.16)}$$

For small $A\sigma$, the resulting "potential well" is almost a square well, and we derive the eigenvalue

$$A\sigma \approx (4D - 1)\exp[-2\pi/(D - \tfrac{1}{4})^{\frac{1}{2}}]. \qquad \text{(E.17)}$$

Thus, for small $D - \frac{1}{4}$, $A\sigma$ is extremely small. The growth rate p does not effectively become of order S until $D \sim \frac{1}{2}$.

For large $A\sigma$, Eq. (16) reduces to the harmonic oscillator equation, and we have the eigenvalue

$$A\sigma = \tfrac{1}{4} + (D - \tfrac{1}{2})^2. \qquad \text{(E.18)}$$

For $G(F')^{-2} \gg \frac{1}{2}$, this may be written

$$p^2/S^2 = G - (F')^2[G/(F')^2 - \tfrac{1}{2}]^{\frac{1}{2}}. \qquad \text{(E.19)}$$

Note that, since we have assumed $|\mu_1| < 1$ in deriving Eqs. (E.1) and (E.2), the discussion of the high-G case is valid only for $\alpha > 1$. Plasma compressibility, which has been neglected throughout, presumably also affects the high-G case.

APPENDIX F. EFFECT OF THERMAL CONDUCTIVITY

From Eq. (5a) we obtain to first order

$$\omega\eta_1 + \mathbf{v}\cdot\nabla\eta_0 = +(2\kappa/3n_0B_0^2)$$
$$\cdot[\mathbf{B}_0\cdot\nabla(\mathbf{B}_0\cdot\nabla\eta_1) + \mathbf{B}_0\cdot\nabla(\mathbf{B}_1\cdot\nabla\eta_0)] \qquad (F.1)$$

so that approximately (setting $B_0^2 = B^2$),

$$\eta_1(p + KF^2\alpha^2) + (i/\alpha)\eta_0'(W - KF\alpha^2\psi) = 0, \qquad (F.2)$$

$$K = 2\kappa\tau_R/3na^2 = 8\pi\kappa/3n\langle\eta\rangle, \qquad (F.3)$$
$$\approx (10^{14}/n)T^4,$$

where T is in eV.

Note that outside the small skin depth of thickness ϵ, around the point where F vanishes, Eq. (19) tells us that $\psi = -WF/p$, so that the correction terms cancel in Eq. (F.2). This is reasonable since in the outside region material is moving with the field lines, and the condition $\mathbf{B}\cdot\nabla T = 0$ is maintained.

If we were to use Eq. (F.2) in the calculations of Sec. V, only the terms proportional to η' would be affected. In the limit $K \to \infty$ we could simply set $\tilde{\eta}' = 0$ in Eq. (39). This would not alter the results for the gravitational and "tearing" modes. For our previous results on the "rippling" mode to be valid, we should require

$$KF'^2\epsilon^2\alpha^2/p < 1 \qquad (F.4)$$

where we have set $F = F'\epsilon$, its value at the edge of the region of the discontinuity. Using Eqs. (34) and (50) and putting $\tilde{\eta}'$, F', $\tilde{\eta}$, $\tilde{p} \approx 1$ we obtain as a condition of validity

$$K(\alpha^{4/5}/S^{6/5}) < 1. \qquad (F.5)$$

Since $K \sim T^4$, $S \sim T^2$, the correction term evidently becomes dominant as $T \to \infty$. In cases of practical interest,[3,31] the critical value of T is of order 10 eV. At higher temperatures a mode of the "rippling" type still exists, but its growth rate depends on K and is greatly diminished relative to that of the ordinary "rippling" mode.

We have assumed here that the classical values of η and κ may be used. In experimental situations where η is enhanced by cooperative phenomena, the magnitude of K may differ from the estimate given in Eq. (F.3).

APPENDIX G. STABILIZATION BY CONDUCTING WALLS

Whether the end-points μ_1, μ_2 are located at finite or infinite μ is important only for low-α modes, in particular for the "tearing" mode. In this appendix we derive a marginal stability condition for the "tearing" mode in the presence of conducting walls located at μ_1, μ_2.

From Eq. (49), we have that the marginal stability condition is characterized by $\Delta' = 0$. From Eqs. (20) and (21) we see then that the stability condition is equivalent to the requirement that $\alpha^2 > \alpha_c^2$, where α_c^2 is the eigenvalue of the equation

$$\psi'' - \psi(\alpha_c^2 + F''/F) = 0 \qquad (G.1)$$

with $\psi = 0$ at μ_1, μ_2. As $|\mu_1|$, $|\mu_2|$ become smaller, we have $\alpha_c^2 \to 0$, and the current layer is then completely stable against the "tearing" mode.

We begin by considering the simple model $F = \tanh\mu$, for which $\alpha_c = 1$ when $\mu_1, \mu_2 = \mp\infty$ (cf. Eq. 28). Equation (G.1) then becomes

$$\psi'' - \psi(\alpha_c^2 - 2/\cosh^2\mu) = 0. \qquad (G.2)$$

The solutions of this equation have been discussed in reference 18. For $\mu_1 = -\mu_2$, one finds $\alpha_c = 0$, 0.50, 0.95 when $\mu_2 = 1.20$, 1.36, 2.20. For $\mu_2 < 1.20$, absolute stability is achieved.

For the general symmetric layer we restrict ourselves to writing down the value of μ_2 which completely stabilizes the tearing mode. In this case $\alpha_c = 0$ and Eq. (G.1) is trivially soluble to give

$$\int_0^{\mu_2} \frac{F''}{FF'^2}\,d\mu + \frac{1}{F(\mu_2)F'(\mu_2)} = 0. \qquad (G.3)$$

The generalization to cylindrical geometry is discussed in Appendix I.

APPENDIX H. EFFECT OF FINITE VISCOSITY

We will consider the case of isotropic fluid viscosity, simply adding a term $\rho\nu\nabla^2\mathbf{v}$ to the inertial term $\rho\, d\mathbf{v}/dt$ in the equation of motion [Eq. (4)]. This treatment indicates the general character and magnitude of viscous effects, but gives only a first approximation to the case of a hot plasma, which is well known to have an extremely complicated viscosity tensor. We will defer consideration of the full viscosity tensor and the Hall-effect terms in Ohm's law (which correspond to finite-Larmor-radius effects) to a later paper, where the present instabilities are approached from the point of view of the full set of plasma equations.

The most appropriate value of ν for the isotropic-viscosity analysis is probably that corresponding to motion transverse to magnetic-field lines. In the case of the modes of Sec. V, this motion exhibits extremely steep transverse velocity gradients in the region of discontinuity near $F = 0$. For the purpose of estimating the order of magnitude of ν in a plasma, we note then that

$$\nu \approx \tau_i v_i^2/(1 + \omega_i^2 \tau_i^2),$$

where v_i, ω_i, and τ_i are respectively the ion thermal velocity, gyrofrequency, and collision time. We are usually interested in the case of singly charged ions and $\omega_i \tau_i \gg 1$. For comparison purposes, the resistivity may be written

$$\eta \approx m_e c^2/ne^2 \tau_e$$

so that

$$\frac{\nu}{\eta} \approx \frac{\beta_i}{4\pi} \frac{\tau_e m_i}{\tau_i m_e}, \qquad \text{(H.1)}$$

where β_i is the ratio of the ion thermal pressure to the magnetic pressure, and m_i, m_e refer to the ion and electron masses. For a fully ionized plasma, we have $\tau_e/\tau_i \approx (m_e/m_i)^{\frac{1}{2}}(T_e/T_i)^{\frac{3}{2}}$. In what follows, we will use the expression $\bar{\rho}\nu = [\langle\eta\rangle/4\pi]q$, where q is generally slightly larger than unity, except for very-low-β or low-(T_e/T_i) plasmas, when it is small, or for $|B_0|$ very small, in which case q may become very large.

Using the modified form of Eq. (4), we now obtain instead of Eq. (14)

$$\bar{\rho}W'' - \frac{q}{p} W'''' = \alpha^2 W\left[\bar{\rho} - \frac{S^2 G}{p^2} + \frac{FS^2}{p} \cdot \left(\frac{F}{\bar{\eta}} + \frac{\bar{\eta}' F'}{\bar{\eta} p}\right)\right] + \psi \alpha^2 S^2 \left(\frac{F}{\bar{\eta}} - \frac{F''}{p}\right). \qquad \text{(H.2)}$$

We have retained only the highest derivatives of W in the inertial and viscous terms. As has been noted previously, for $S \to \infty$ the left-hand side of Eq. (14) is important only in the region of discontinuity. Since $q/p \to 0$ in the high-S limit, this remark is equally true for Eq. (H.2). The predominance of the highest derivatives of W follows from the same consideration.

To calculate the effect of viscosity in the region of discontinuity, we may proceed as in Sec. V, using Eqs. (13) and (H.2). We note that the viscous term is of order $q/\bar{\rho}p\epsilon^2$ relative to the inertial term. Therefore (unless $q \ll 1$) the viscous term will predominate, and this is the situation that we will consider here. Since the $\bar{\rho}W''$ term is negligible except in the limit of Appendix C, we note that the mass density now completely disappears from the equations, and is replaced by an "effective mass density" $\rho_\nu = q/p\epsilon^2$. Thus we may adapt the analysis of Sec. V simply by replacing $\bar{\rho}$ with ρ_ν. The basic scale unit ϵ of Eq. (34) now becomes

$$\epsilon = (q^{\frac{1}{6}}\bar{\eta}^{\frac{1}{6}}/2\alpha S\,|F'|)^{\frac{1}{3}} \qquad \text{(H.3)}$$

so that

$$\rho_\nu = (1/p)(2\alpha S\,|F'|\,q/\bar{\eta}^{\frac{1}{2}})^{\frac{2}{3}}. \qquad \text{(H.4)}$$

Equations (31) and (32) are unaltered except in the interpretation of ϵ, and Eq. (33) becomes

$$U'''' - U(\Lambda - \tfrac{1}{4}\theta^2) = -\psi(\theta - \delta). \qquad \text{(H.5)}$$

We note that the homogeneous equation

$$U'''' - U(\Lambda - \tfrac{1}{4}\theta^2) = 0$$

is derived from the variational form

$$\Lambda = \int_{-\infty}^{\infty} d\theta\, [(U'')^2 + \tfrac{1}{4}\theta^2 U^2] \Big/ \int_{-\infty}^{\infty} d\theta\, U^2 \qquad \text{(H.6)}$$

so that there is a set of positive eigenvalues Λ, with a lowest eigenvalue of order unity. Thus the solution of Eqs. (32) and (H.5) proceeds in a manner very similar to the solution of Eqs. (32) and (33). In the growth rates of Eqs. (50), (57), and (59), we may simply replace $\bar{\rho}$ by ρ_ν and obtain approximately for the "rippling" mode,

$$p \approx \frac{|\bar{\eta}'|}{3\Lambda^{\frac{1}{2}}}\left(\frac{\alpha S\,|F'|}{q^{\frac{1}{2}}\bar{\eta}^{\frac{1}{2}}}\right)^{\frac{1}{2}}, \qquad \text{(H.7)}$$

for the "tearing" mode,

$$p \approx \frac{1}{3}\left(\frac{2S\bar{\eta}^{-5/2}\,|F'|^7}{\alpha^2 q^{\frac{1}{2}}}\right)^{\frac{1}{6}}\left(\frac{1}{F_{-\infty}^2} + \frac{1}{F_\infty^2}\right), \qquad \text{(H.8)}$$

and for the gravitational interchange mode

$$p \approx \frac{G}{\Lambda}\left(\frac{\alpha S\bar{\eta}}{4\,|F'|^2\,q^{\frac{1}{2}}}\right)^{\frac{2}{3}}, \qquad \text{(H.9)}$$

where the fastest growing modes have $\Lambda = O(1)$.

We note that our previous results are left qualitatively unaltered. For the "rippling" and "tearing" modes, the effective mass density ρ_ν becomes large as $S \to \infty$, and the thickness ϵ of the region of discontinuity then increases, while the growth rates are depressed somewhat. For the gravitational interchange mode, we have

$$\rho_\nu = (q\Lambda/\bar{\eta}G)(4F')^{4/3}, \qquad \text{(H.10)}$$

which is independent of S. Therefore ϵ and p are altered only by constant factors.

APPENDIX I. EFFECTS OF CYLINDRICAL GEOMETRY

When applied to nonplanar current layers, the stability analysis of the plane resistive current layer must be extended in two major aspects.

(1) Allowance must be made for the destabilizing force associated with a negative plasma-pressure gradient along the radius of curvature; this effect has been simulated approximately in the planar analysis by means of a gravitational field, but an exact interpretation of G remains to be given.

(2) For long-wave modes, i.e., particularly for the "tearing" mode, which is always long-wave in the present sense, the cylindrical geometry modifies the solution in the "outer region," and therefore affects the value of Δ' to be used in the dispersion relation. We will treat the usual high-S limit.

Interpretation of G

To generalize the planar analysis, we will use a cylindrical coordinate system where $x \to r\theta$, $y \to r$, and $z \to z$. Thus the zero-order configuration is given by

$$\mathbf{B}_0 = \hat{\theta} B_{\theta 0}(r) + \hat{z} B_{z0}(r) \tag{I.1}$$

and the perturbations are given by

$$f_1(\mathbf{r}, t) = f_1(r) \exp\,[i(m\theta + k_z z) + \omega t].$$

In analogy with quantities defined in Sec. II and Appendix A, we will write

$$\psi = B_{r1}/B, \qquad W = -iv_{r1}k\tau_R,$$
$$\chi = (ia/B)[k_z B_{\theta 1} - (m/r)B_{z1}],$$
$$V = \tau_R[k_z v_{\theta 1} - (m/r)v_{z1}],$$
$$F = [(m/r)B_{\theta 0} + k_z B_{z0}]/kB,$$
$$H = [k_z B_{\theta 0} - (m/r)B_{z0}]/kB,$$
$$k = (k_z^2 + m^2/R^2)^{\frac{1}{2}},$$

where R is the radius of $F = 0$. We will use $\tilde{k}_z = k_z/k$, $\tilde{R} = R/a$.

As in the planar analysis the effect of the destabilizing mechanism appears only in a small region $r \approx R$, and for convenience we will specialize our equations to hold in this region. Since we are not concerned here with the "rippling" and "tearing" modes, we may neglect the η_1 and F'' terms in what follows. Thus we obtain from the pressure-balance equation [Eq. (9)]

$$(p\bar{p}/\alpha^2 S^2)W'' = F\psi'' - 2\tilde{k}_z(H_\theta/\tilde{R})\chi, \tag{I.2}$$

where $H_\theta = B_{\theta 0}/B$. The independent variable is $\mu = r/a$. In deriving Eq. (I.2), we have made the usual approximation (cf. Sec. V) that for zero-order quantities $\epsilon f_0' \ll f_0$. In the present context, this includes $\epsilon \ll \tilde{R}$. We have also neglected terms in $\alpha\epsilon$, which was found to be appropriate in Sec. V. Finally, we have used $\psi' \ll \chi$, which is justified in Appendix A. [From the latter remark and from Eq. (10), it follows incidentally that $\mathbf{k}\cdot\mathbf{B}_1 = 0$ in the region of discontinuity, so that $B_{\theta 1}$, B_{z1}, and χ all have the same r-dependence.] From Ohm's Law (Eq. 8), we obtain

$$\psi'' = (p/\bar{\eta})\psi + (F/\bar{\eta})W \tag{I.3}$$

as before, and we now need the additional component

$$\chi'' = \frac{p}{\bar{\eta}}\chi + \frac{\alpha F}{\bar{\eta}} V - \frac{W}{\bar{\eta}}\left[\tilde{k}_z\left(H_\theta' - \frac{H_\theta}{\tilde{R}}\right) - \frac{m}{\tilde{R}} H_z'\right]. \tag{I.4}$$

Equation (I.4) in turn introduces the dimensionless velocity component V in the $\mathbf{k}\times\mathbf{B}$ direction, so that we must make use of the appropriate component of the pressure-balance equation, and obtain

$$\frac{p\bar{p}}{S^2} V = \alpha F\chi + \psi\left[\tilde{k}_z\alpha\left(H_\theta' + \frac{H_\theta}{\tilde{R}}\right) - \frac{m}{\tilde{R}} H_z'\right]. \tag{I.5}$$

(The χ and W terms of Eq. (I.4) are now seen to be of order $p^2/\alpha^2 S^2 \epsilon^2 (F')^2 \sim p^{\frac{3}{2}}/\alpha S F' \sim G/(F')^2 \ll 1$ [cf. Eq. (59)] relative to the V term, and are negligible except in the special limit of Case 2, discussed below.)

Using Eqs. (I.2–I.5), we may now carry out an expansion procedure like that of Sec. V using

$$\theta_1 = (\mu - \tilde{R})/\epsilon.$$

It is convenient to introduce $\beta_0 = 4\pi P_0/B^2$, for which the zero-order pressure-balance equation gives

$$\beta_0' + H_\theta H_\theta' + H_\theta^2/\tilde{R} + H_z H_z' = 0. \tag{I.6}$$

We also use $\chi = \Omega X$, where $\Omega = \epsilon p/4\bar{\eta}$, and we recall $U = W4\epsilon F'/p$. The equations may then be written

$$\frac{d^2U}{d\theta_1^2} - \tfrac{1}{4}\theta_1^2 U = \theta_1\psi - \frac{\tilde{k}_z H_\theta}{2\tilde{R}F'} X, \tag{I.7}$$

$$d^2\psi/d\theta_1^2 = \epsilon\Omega(4\psi + \theta_1 U), \tag{I.8}$$

$$\frac{d^2X}{d\theta_1^2} - X\left(\frac{\theta_1^2}{4} + 4\epsilon\Omega\right) = -\theta_1 \frac{\tilde{k}_z\beta_0'}{4\epsilon\Omega F' H_\theta}\psi + \frac{\tilde{k}_z(\beta_0' + 2H_\theta^2/\tilde{R})}{F' H_\theta} U. \tag{I.9}$$

It is of incidental interest to note that F' is related to the "magnetic shear" by the equation

$$F' = -\tilde{k}_z H_z[\log(H_\theta/\mu H_z)]'.$$

We next eliminate X from Eqs. (I.7) and (I.9), obtaining a fourth-order equation, and we solve by expanding U as in Eq. (40). We obtain

$$a_n\left[(n + \tfrac{1}{2})(n + \tfrac{1}{2} + 4\epsilon\Omega) + \frac{\tilde{k}_z^2(\beta_0' + 2H_\theta^2/\tilde{R}}{2\tilde{R}(F')^2}\right] = \left[\frac{\tilde{k}_z^2\beta_0'}{8\tilde{R}(F')^2\epsilon\Omega} - (n + \tfrac{1}{2} + 4\epsilon\Omega)\right]\int_{-\infty}^{\infty} d\theta_1\, \theta_1 u_n \psi. \qquad (I.10)$$

Making the constant-ψ approximation, as in section V, and using the integrals listed below Eq. (46), we obtain

$$\Delta' = 2^{3/2}\Omega \sum_{m=0}^{\infty}\left\{\frac{m + \frac{3}{4} + 2\epsilon\Omega + [\tilde{k}_z^2/4\epsilon\Omega\tilde{R}(F')^2][(m + \frac{1}{2})\beta_0' + 2\epsilon\Omega(\beta_0' + 2H_\theta^2/\tilde{R})]}{(m + \frac{3}{4})(m + \frac{3}{4} + 2\epsilon\Omega) + [\tilde{k}_z^2/8\tilde{R}(F')^2](\beta_0' + 2H_\theta^2/R)}\right\}\frac{\Gamma(m + \frac{1}{2})}{\Gamma(m + 1)}. \qquad (I.11)$$

We will be interested mainly in short-wave instabilities, so that, as in the plane case, we have $\Delta' = -2\alpha$, where $\alpha \gg 1$. We note immediately that $\beta_0' \geq 0$ is sufficient for stability. (If $B_{\theta 0} \equiv 0$, then the point $F = 0$ occurs at $\beta_0' = 0$, so that there is no instability.) We note also that $\epsilon\Omega$ cannot be greater than of order $\beta_0'/(F')^2$, if the right-hand side of Eq. (I.11) is to be negative. Thus we may neglect $2\epsilon\Omega$ relative to $m + \frac{3}{4}$.

We now obtain the growth rate for $\beta_0' < 0$. Since $\epsilon \to 0$ as $S \to \infty$, the β_0'/ϵ term in the numerator will predominate on the right-hand side of Eq. (I.11), unless Ω becomes large. Thus the finiteness of Δ' implies $\Omega \to \infty$, which in turn implies that the sum of the series in Eq. (I.11) goes to zero.

As has been noted in Sec. V, in the present analysis of finite-conductivity interchange modes of the gravitational type, we must restrict ourselves to a range of parameters such that the infinite-conductivity modes are stable. Thus the Suydam criterion, $8\beta_0'\tilde{k}_z^2/\tilde{R}(F')^2 < 1$, must hold. In that case, the β_0'-contribution to the $(\beta_0' + 2H_\theta^2/\tilde{R})$- terms in Eq. (I.11) is seen to be negligible relative to the $(m + \frac{3}{4})$ terms. The $(2H_\theta^2/\tilde{R})$ contribution is also negligible when $(H_\theta\tilde{k}_z/\tilde{R}F')^2 \ll 1$. This is the usual case, which we will refer to as Case 1. The opposite condition is satisfied for Case 2.

Case 1 is the case of large shear. This appears more clearly if we write the defining condition as

$$[\log (H_\theta/\mu H_z)]'^2 \gg (H_\theta/\mu H_z)^2. \qquad (I.12)$$

In order that the series should be near a null we require

$$\tilde{k}_z^2\beta_0'/\epsilon\Omega\tilde{R}(F')^2 \approx 5. \qquad (I.13)$$

Evidently there is only a single null, corresponding to the growth rate

$$p = (2S\alpha\tilde{k}_z^2\, |\beta_0'|\, \tilde{\eta}^{1/3}/5\, |F'|\, \tilde{R}\tilde{\rho}^{1/2})^{2/3}. \qquad (I.14)$$

Thus we may identify the quantity G of Eq. (59) somewhat loosely with $-\tilde{k}_z^2\beta_0'/\tilde{R}$. We note, however, that the pressure-gradient-destabilization term in Eq. (I.7) is not effectively identical with the gravitational-force term in Eq. (33)—for example, it gives rise to only a single unstable mode instead of to a whole spectrum. Also, allowing for finite compressibility does not affect the present result, while we have seen in Appendix A that the true gravitational mode is somewhat modified.

Case 2 is the case of small shear, where the opposite of Eq. (I.12) holds. If the Suydam criterion is to be satisfied also, Case 2 can occur only for $8\, |\beta_0'| \ll H_\theta^2/\tilde{R}$. From Eq. (I.11), one estimates then that

$$p \sim (S\alpha\tilde{k}_z\, |\beta_0'|\, \tilde{\eta}^{1/3}/H_\theta\tilde{\rho}^{1/2})^{2/3}. \qquad (I.15)$$

The Tearing Mode

Only the solution in the "outer region" is affected by cylindrical geometry. From Eq. (9), with $\mathbf{v} \equiv 0$, one obtains a second-order differential equation for ψ, similar to Eq. (20), but somewhat less tractable. Given m and α_z, one may calculate Δ' as in Sec. III; or else one may set $\Delta' = 0$, as in Appendix G, and obtain a stability condition on m and α_z. To do the complete analysis, goes beyond the scope of this paper, but several points perhaps deserve comment.

1. Except for the $m = 0$ mode, the quantity α can no longer be made arbitrarily small, since

$$\alpha^2 = \frac{m^2}{\tilde{R}^2}\left(1 + \left.\frac{B_{\theta 0}^2}{B_{z0}^2}\right|_{\mu_0}\right). \qquad (I.16)$$

From the plane results (Appendix G) we know that small α is most unstable, and similarly we expect small m and large $\tilde{R}$ to be most unstable here. We note also that large $B_{\theta 0}/B_{z0}$ prevents low-α modes for $m > 0$.

2. A plausible approximation[18] is to treat the layer itself as being approximately plane ($a \ll R$), so that Eq. (20) applies, and to use the familiar Bessel-function solutions in the vacuum regions. This is a useful method for proving instability in the case of the more unstable configurations, (for example, most "stabilized pinches"). From the point of view of obtaining exact stability criteria, this approach is unfortunately not wholly satisfactory, since one finds that stability *cannot* be achieved under the conditions where the approximation is both valid and useful. That is to say, stability requires either $a \sim R$, or else $B_{\theta 0}/B_{z0}\, |_{\mu_0} \sim R/a$, (for either of which Eq. (20) is inadequate); or else,

there must be close-fitting conducting walls, so that the plane approximation holds in the vacuum region also, if it holds in the current layer itself.

Note added in proof. The marginal stability problem for the "tearing" mode in cylindrical geometry is closely related to the problem of "neighboring equilibria" investigated by Rebut.[37] The neighboring-equilibrium analysis, however, necessarily confines itself to $P_0' = 0$ at $\mu = \mu_0$, whereas we have seen that the "tearing" mode exists more generally. If $P_0' \neq 0$ at $\mu = \mu_0$, then χ becomes discontinuous as $p \rightarrow 0$; but for large p there is no such difficulty. Further results on the cylindrical stability problem with $P_0' = 0$ at $\mu_0 = 0$ are given in references 38 and 39.

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INSTABILITY OF THE POSITIVE COLUMN IN A MAGNETIC FIELD AND THE 'ANOMALOUS' DIFFUSION EFFECT*

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Abstract—The positive column of a gas discharge is shown to become unstable when a sufficiently large longitudinal magnetic field is superimposed. This instability causes oscillations which lead to an increase in the flux of charged particles striking the walls; that is, in agreement with the experimental results, there is an increase in the effective diffusion.

1. INTRODUCTION

THE diffusion of electrons and ions across a magnetic field is a problem which is of considerable interest to plasma physics but it is one which up to the present time has yielded some very conflicting experimental results (GUTHRIE and WAKERLING, 1949; SIMON, 1955; BOSTICK and LEVIN, 1955; NEDOSPASOV, 1958; LEHNERT, 1959; HOH and LEHNERT, 1959; ZHARINOV, 1959; ELLIS *et al.*, 1959; SIRGII and GRANOVSKII, 1959 and VASIL'EVA and GRANOVSKII, 1959). In addition, in a number of experiments (GUTHRIE and WAKERLING, 1949; BOSTICK and LEVIN, 1955; LEHNERT, 1959; HOH and LEHNERT, 1959; ZHARINOV, 1959 and ELLIS *et al.*, 1959). an anomalously large diffusion of the plasma across the magnetic field has been observed. Consequently, experiments on the diffusion of the plasma in the positive column of a gas discharge are of special interest and in this connexion we would particularly cite the work of LEHNERT and HOH (1959) and also LEHNERT (1959). In the absence of an applied magnetic field this type of discharge is well understood both theoretically and experimentally. It therefore forms a convenient subject for theoretical treatment and for comparison with experiment. It is to this problem that the work reported here was devoted.

LEHNERT and HOH have measured the electric field E along the positive column as a function of the magnitude of the longitudinal magnetic field H. With magnetic fields which are not too strong they found that the electric field falls off as H increases in quantitative agreement with the theoretical form of the dependence of the ambipolar diffusion coefficient on the magnetic field strength. However, at a certain critical field H_c, which is of the order of several kilogauss, the character of the dependence of E on H was found to undergo a sudden change; the electric field increased with H and in some cases it actually exceeded the value at $H = 0$. This increase in E provides indirect evidence that the diffusion coefficient is considerably larger than the 'classical' value derived from a consideration of collisions between the electrons and ions with neutral gas molecules.

We show below that this effect can be explained by the instability of the positive column in the longitudinal magnetic field and by the development of oscillations of the 'diffusion' type.

2. INSTABILITY OF THE POSITIVE COLUMN

One can see from a simple qualitative argument that a longitudinal magnetic field can lead to a loss of stability of the plasma current channel. In the absence of a magnetic field any disturbance of the plasma will be rapidly terminated by the increased diffusion of particles from regions of excess density. In particular, the twisting and wriggling of the current channel causes an increase in the flux of particles striking the wall at those places where the channel comes close to it and a decrease in the flux on the reverse side. This has the effect of restoring the original state of the plasma. In the presence of a magnetic field there is an additional force $(1/c)\mathbf{j} \times \mathbf{H}$ acting on the plasma, where $\mathbf{j}$ is the current density. With a helical distortion of the plasma this force acts either towards the axis of the discharge or towards the walls according to whether the helix is right or left-handed. When this force acts in the direction of the walls there is a build-up of the initial disturbance. As the magnetic field is increased this force becomes greater and the transverse diffusion, and along with it, the stabilizing action of the walls is reduced. Thus, for sufficiently large magnetic fields the plasma column becomes unstable.

In order to investigate this stability problem we will make use of the diffusion equations for electrons and ions:

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_i = \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v}_e = Zn. \qquad (1)$$

* A translation by D. L. ALLAN of this paper originally submitted in Russian.

Here n is the electron (and also the ion) density.

$Z = Z(T_e)$ is the number of ionization events per electron per unit time, T_e is the electron temperature, $\mathbf{v}_e$, $\mathbf{v}_i$ are the directional velocities of the electrons and ions which may be found from the equations of motion.

For the electrons, the equation of motion has the form:

$$\frac{kT_e}{mn}\nabla n = -\frac{e}{mc}\mathbf{v}_e \times \mathbf{H} + \frac{e}{m}\nabla V - \frac{\mathbf{v}_e}{\tau} \qquad (2)$$

where m is the electron mass, $1/\tau$ is the collision frequency and V is the electric field potential. With $\Omega\tau = (eH/mc)\tau \gg 1$, we obtain from equation (2):

$$v_{e\perp} = \frac{b_e}{\Omega\tau}\mathbf{h} \times \nabla V - \frac{D_e}{\Omega\tau}\frac{1}{n}\mathbf{h} \times \nabla n + \frac{b_e}{(\Omega\tau)^2}\nabla V - \frac{D_e}{(\Omega\tau)^2}\frac{\nabla n}{n}$$

$$v_{ez} = b_e\frac{\partial V}{\partial z} - \frac{D_e}{n}\frac{\partial n}{\partial z} \qquad (3)$$

where $\mathbf{h} = \mathbf{H}/H$ is the unit vector along the z-axis, b_e is the electron mobility and D_e is the electron diffusion coefficient.

We will further suppose that the frequency of the collisions between the ions and the neutral gas molecules $(1/\tau_i)$ is much greater than the cyclotron frequency eH/Mc and also much greater than the frequency of the oscillations we are considering. With this assumption we can neglect the effect of the magnetic field on the ions and also the inertia of the ions. We then obtain:

$$\mathbf{v}_i = -b_i\nabla V \qquad (4)$$

where b_i is the mobility of the ions. We have neglected here the diffusion of the ions. It can be shown that taking the diffusion of the ions into account introduces only a small correction of the order of the ratio of the ion temperature to the electron temperature.

Substituting the expressions for the velocities (3) and (4) into (1), we obtain two equations for n and V:

$$\frac{\partial n}{\partial t} + \frac{b_e}{\Omega\tau}\mathbf{h}\,.\,\nabla V \times \nabla n - \frac{D_e}{(\Omega\tau)^2}\Delta_\perp n + \frac{b_e}{(\Omega\tau)^2} \times \operatorname{div} n\nabla_\perp V + \frac{\partial}{\partial z}\left(b_e n\frac{\partial V}{\partial z} - D_e\frac{\partial n}{\partial z}\right) = Zn \qquad (5)$$

$$\frac{\partial n}{\partial t} - b_i \operatorname{div} n\nabla V = Zn. \qquad (6)$$

In the equilibrium state n and V depend only on distance from the axis of the discharge tube. We will suppose that the mean-free-path of the ions λ_i is much less than the radius of the tube a. Then, as a boundary condition, we can equate the density at the wall to zero (Shottki condition), whereupon from (5) and (6) we obtain

$$n_0(r) = N_0J_0(\beta_0 r), \qquad \frac{dV_0}{dr} = \frac{D_e}{b_e + b_i(\Omega\tau)^2}\frac{1}{n_0}\frac{dn_0}{dr}$$

$$Z = \frac{b_i}{b_e}\frac{D_e\beta_0^2}{1 + \frac{b_i}{b_e}(\Omega\tau)^2} \qquad (7)$$

Here $\beta_0 = (\alpha_0/a)$, α_0 is the first root of the zero order Bessel function J_0, N_0 is the density at $r = 0$.

In order to obtain the stability conditions we must find the frequency of the small oscillations of of the current channel. Because the equilibrium state has cylindrical symmetry we can choose a perturbation of the form $f(r)\exp(im\psi + ikz - i\omega t)$ where ψ is the azimuthal angle. The linearized equations have the form:

$$\left\{-i\omega - Z + ikv_0 + k^2D_e + \frac{im}{r}\frac{b_e}{\Omega\tau}\frac{dV_0}{dr}\right\}n' - \frac{D_e}{(\Omega\tau)^2}\Delta_\perp n' + \frac{b_e}{(\Omega\tau)^2}\frac{1}{r}\frac{d}{dr}\left(rn'\frac{dV_0}{dr}\right) - \left\{k^2b_en_0 + \frac{im}{r}\frac{b_e}{\Omega\tau}\frac{dn_0}{dr} + \frac{m^2}{r^2}\frac{b_en_0}{(\Omega\tau)^2}\right\}V' + \frac{b_e}{(\Omega\tau)^2}\frac{1}{r}\frac{d}{dr}\left(rn_0\frac{dV'}{dr}\right) = 0 \qquad (8)$$

$$(-i\omega - Z)n' + b_ik^2n_0V' + b_i\frac{m^2}{r^2}n_0V' - b_i\frac{1}{r}\frac{d}{dr}r\left(n_0\frac{dV'}{dr} + n'\frac{dV_0}{dr}\right) = 0. \qquad (9)$$

Here $v_0 = b_eE$ is the directional (current) velocity of the electrons in the equilibrium state and E is the longitudinal electric field.

To the equations (8) and (9) we must now add the boundary conditions. The first two boundary conditions imply that n' and V' are regular at $r = 0$, the third condition is that the density at the wall is zero and the fourth follows from the equality of the electron and ion currents to the wall. Since $(1/n)(\partial n/\partial r) \sim (1/\lambda_i)$ is a constant at the boundary, then the last condition is satisfied if the potential V' is finite at the wall.

An accurate treatment of the set of equations (8), (9) is rather difficult, and therefore we will find an approximate solution, specifying a certain radial dependence for n' and V'. We will confine ourselves here to perturbations for which $|m| = 1$, since such perturbations will define the region of stability. It is also natural to choose a density dependence of the form $n' = n_1 J_1(\beta_1 r)$, where $\beta_1 = (\alpha_1/a)$ and α_1 is the first root of the Bessel function J_1. We will adopt the same form for V', that is $V' = V_1 J_1(\beta_1 r)$; the reason for this choice will be apparent later.

Let us substitute these expressions into equations (8) and (9), multiply them by $J_1(\beta_1 r) r\, dr$ and integrate with respect to r. We then obtain a set of two algebraic equations for n_1 and V_1:

$$\left\{-i\omega - Z + ikv_0 + k^2 D_e + \frac{D_e \beta_1^2}{(\Omega\tau)^2} - im\frac{L}{\Omega\tau} \times \right.$$

$$\left. \times \frac{D_e\beta_0^2}{1 + \frac{b_i}{b_e}(\Omega\tau)^2} - \frac{Q}{(\Omega\tau)^2}\frac{D_e\beta_0^2}{1 + \frac{b_i}{b_e}(\Omega\tau)^2}\right\}\frac{n_1}{n^0}$$

$$- \left\{k^2 b_e - imA\beta_0^2\frac{b_e}{\Omega\tau} + C\frac{b_e\beta_1^2}{(\Omega\tau)^2}\right\}V_1 = 0 \qquad (10)$$

$$\left\{-i\omega - Z + \frac{b_i}{b_e}\frac{QD_e\beta_0^2}{1 + \frac{b_i}{b_e}(\Omega\tau)^2}\right\}\frac{n_1}{n^0} + Cb_i\beta_1^2 V_1 = 0 \qquad (11)$$

Here n^0 is some average value of the unperturbed density and $A = 0{\cdot}66$, $C = 0{\cdot}79$, $L = 0{\cdot}74$ and $Q = 1{\cdot}56$ are the constants appearing after integrating.

In equation (11), we have neglected the motion of the ions along the z-axis in view of the fact that $b_i/b_e \ll 1$.

Putting the determinant of the set (10), (11) equal to zero we readily obtain the dispersion equation for ω. We have a stable condition only when the imaginary part of the frequency is negative. Using the dispersion equation, this condition can be written in the form:

$$KX^4 + FX^2 + G \geqslant mBv^* X\frac{b_e}{b_i} \qquad (12)$$

where

$$K = \frac{1{\cdot}28 + y}{y(y+1)}; \qquad F = \frac{0{\cdot}8\,(y+2)}{y};$$

$$G = \frac{0{\cdot}48(1+y)}{y} + 0{\cdot}1\frac{b_e}{b_i}\frac{1}{1+y} \qquad (13)$$

$$B = 0{\cdot}163; \quad X = \frac{k\Omega\tau}{\beta_1}; \quad v^* = \frac{v_0}{\beta_0 D_e}; \quad y = \frac{b_i}{b_e}(\Omega\tau)^2$$

From relation (12) it can be seen that the instability arises only when $m > 0$ and when the electron drift velocity v_0 is sufficiently large. The plasma is stable when there is no longitudinal current.

In a long tube arbitrary values X are possible. Therefore the plasma becomes unstable when the left-hand part of (12) turns out to be equal to the right-hand part at a unique point. In this case the derivatives of both parts will be equal too. Differentiating the equality (12) with respect to X, multiplying the result by X and subtracting one from the other we obtain a biquadratic equation for X, whence:

$$X^2 = \frac{-F + \sqrt{F^2 + 12KG}}{6K},$$

$$v^* = \frac{2b_i}{Bb_e}X(2KX^2 + F). \qquad (14)$$

Relation (14) determines the boundary of the stability region $v^* = v^*(y)$ where $v^* = (b_e/\beta_0)(E/D_e) = (eE/\beta_0 kT_e)$. The electron temperature is a function of E/p and is well-known experimentally (KLARFELD, 1941; KARELINA, 1941) and the same is true also of the mobilities b_i, b_e (ENGEL, 1955; GRANOVSKII, 1952). Using these experimental results we can calculate v^* and b_e/b_i for those values of E for which the monotonic dependence of E upon H has been observed to break down and then, using (14), we can find y and so find a theoretical value for the critical magnetic field H_c.

We carried out this calculation for helium. We took the values of the electric field corresponding to $H = H_c$ from the experimental results of HOH and LEHNERT and the values of T_e were taken from the work of KLARFELD. A comparison of the calculated values H_{ct} with the experimental values H_{ce} is given in Table 1.

A result corresponding to the last row and first column of this table has not been computed, for here the conditions are such that it is not possible to ignore the action of the magnetic field upon the ions.

A similar calculation was carried out for argon but in this case there is only one experimental point: $H_{ce} = 3{\cdot}7$ kgauss with $a = 1$ cm and $p = 0{\cdot}92$ mm Hg. Once again the calculated value of the critical field $H_{ct} = 3{\cdot}3$ kgauss agrees well with the experimental value.

Since all the coefficients in (12) are roughly of the same order of magnitude, X turns out to be of the order of unity, i.e. the instability develops for long wave perturbations $ka = (\alpha_0 X/\Omega\tau) \ll 1$. This explains why, in the experiments of LEHNERT and HOH, the dependence on the length of the tube L disappears only when $L/2a \geqslant 50$. For shorter tubes X is restricted to lower values. Thus, depending on the actual relations between the various parameters in (12) the plasma channel will either become stable again as H increases or it will be stable with respect to

TABLE 1

ap	cm × mm Hg	0·24	0·45	0·89	1·45	1·6	1·8	3·0
a	cm	0·535	1	1	1	0·535	1	1
H_{cr}/p	kgauss/mm Hg	5·5	3·6	2·4	1·6	1·9	1·5	0·9
H_{ct}/p (He^+)	kgauss/mm Hg	5·1	2·8	2·0	1·6	1·5	1·4	1·0
H_{ct}/p (He^+_2)	kgauss/mm Hg	—	3·3	2·1	1·6	1·5	1·4	1·0

perturbations with $m = 1$ at all values of H. In the latter case, the loss of stability will occur when oscillations with $m > 1$ arise.

We will now return to the question of the choice of the radial dependence for V'. Let us multiply equation (9) by $[b_e]/[b_i(\Omega\tau)^2]$ add it to (8) and neglect small terms of the order

$$\frac{b_i}{b_e} y \sim \left(\frac{eH}{Mc}\tau_i\right)^2.$$

Then, by virtue of (7), we obtain:

$$n' \frac{D_e\beta_0^2}{b_e}\left\{\frac{\alpha_1}{\alpha_0} X v^* + \frac{m}{r(1+y)\beta_0^2}\frac{1}{n_0}\frac{dn_0}{dr}\right\} \approx \frac{m}{r}\frac{dn_0}{dr} V'. \tag{15}$$

By making use of the condition (12), one can show that when the plasma becomes unstable the left-hand side of (15) turns out to be everywhere positive except in a small region near the wall where it changes sign and goes to a finite negative value as $r \to a$. Since the density n' near the wall is small, this region makes only a small contribution while for the remaining values of r the function $J_1(\beta_1 r)$ is a satisfactory approximation for V'.

3 'ANOMALOUS' DIFFUSION

The oscillations of the plasma current channel which arise as a result of the instability lead to the appearance of an azimuthal electric field. The electrons are therefore able to drift in a radial direction which, when averaged, appears as an increased mobility across the magnetic field; in other words, we have an 'anomalous' diffusion.

For the purpose of providing a quantitative description of this effect we can no longer use the linearized diffusion equations. Provided that the magnetic field is not much larger than H_c, the amplitude of the oscillations will be small and can be treated as harmonic. Then, as before, we can write

$$n = n_0(r) + n_1 \cos(\psi + kz - \omega t) J_1(\beta_1 r),$$
$$V = V_0(r) + V_1 \cos(\psi + kz - \omega t + \delta) J_1(\beta_1 r) \tag{16}$$

where the phase shift δ can be considered to be approximately independent of r.

Substituting these expressions into equations (5) and (6) and averaging over time, we obtain

$$\frac{1}{r}\frac{d}{dr} r\left\{\frac{b_e}{(\Omega\tau)^2} n_0 \frac{dV_0}{dr} - \frac{D_e}{(\Omega\tau)^2}\frac{dn_0}{dr} + \frac{b_e}{\Omega\tau}\frac{n_1 V_1}{r}\frac{\sin\delta}{2} \times \right.$$
$$\times J_1^2(\beta_1 r) + \frac{b_e}{(\Omega\tau)^2}\beta_1 n_1 V' J_1(\beta_1 r) \times$$
$$\left. \times J_1'(\beta_1 r)\frac{\cos\delta}{2}\right\} = Zn_0 \tag{17}$$

$$-\frac{1}{r}\frac{d}{dr} r\left\{b_i n_0 \frac{dV_0}{dr} + b_i n_1 V_1 \beta_1 J_1(\beta r) \times \right.$$
$$\left. \times J_1'(\beta r)\frac{\cos\delta}{2}\right\} = Zn_0. \tag{18}$$

When the amplitude of the oscillations is small, the approximate solutions of equations (17) and (18) can be found by putting

$$n_0 = N_0 J_0(\beta_0 r), \quad \frac{dV_0}{dr} = \frac{1}{n_0}\frac{dn_0}{dr}\frac{D_e}{b_e(1+y)} S,$$

where $S = \text{const}$.

We will now insert this into equations (17) and (18) multiply them by $J_0(\beta_0 r) r\, dr$ and integrate with respect to r. From the equations obtained we can find Z and S:

$$Z = \frac{b_i}{b_e}\frac{D_e\beta_0^2}{1+y}\phi, \quad S = \phi - \rho\frac{(\phi-1)(1+y)}{\Omega\tau}\operatorname{ctg}\delta. \tag{19}$$

Here $\phi = 1 + \Omega\tau s \,.\, \sin\delta$, s is a small quantity of the order of the amplitude of the oscillations and $\rho = 2{\cdot}6$ is a constant which is expressed in terms of the integrals of the Bessel functions.

In the equations for small oscillations we can neglect the non-linear terms since they contain second harmonics which disappear when averaged over a half-cycle. Thus n' and V' may once more be considered as complex quantities and the only change we have to make to equations (8) and (9) is to replace Z and dV_0/dr by the new values.

Making this change and repeating the calculation, we obtain the following dispersion equation ($m = 1$):

$$\frac{i\omega}{C}\left\{\frac{C(1+y)}{y}+\frac{X^2}{y}-iA\left(\frac{\alpha_0}{\alpha_1}\right)^2\frac{b_e}{b_1\Omega\tau}\right\}=\frac{D_e\beta_1^2}{(\Omega\tau)^2}\times$$

$$\times\left\{i\Omega\tau\frac{\alpha_0}{\alpha_1}v^*X+1-\left(\frac{\alpha_0}{\alpha_1}\right)^2\phi+X^2\left[1+\left(\frac{\alpha_0}{\alpha_1}\right)^2\times\right.\right.$$

$$\left.\left.\times\frac{SQ-\phi}{C(1+y)}\right]-\frac{i\Omega\tau}{1+y}\left[SL+A\left(\frac{\alpha_0}{\alpha_1}\right)^2\frac{SQ-\phi}{C}\right]\right\} \qquad (20)$$

The condition $\text{Im}\ \omega = 0$ takes the form:

$$\frac{1+y+0{\cdot}78S-0{\cdot}5\phi}{y(1+y)}\,.\,X^4+$$

$$+\frac{0{\cdot}8y+1{\cdot}8+0{\cdot}6S-0{\cdot}8\phi}{y}\,.\,X^2+0{\cdot}8(1-0{\cdot}4\phi)\times$$

$$\times\frac{1+y}{y}+0{\cdot}1\frac{b_e}{b_i}\frac{1{\cdot}3S-0{\cdot}3\phi}{1+y}=0{\cdot}163\frac{b_e}{b_i}Xv^* \qquad (21)$$

It then follows that the frequency of the oscillations given by equation (20) is approximately

$$\omega\approx\frac{b_i}{b_e}\frac{3D_e\beta_1^2}{\Omega\tau}\{1-0{\cdot}4\phi+X^2\} \qquad (22)$$

From the equation for the ions it is not difficult to find the phase shift δ. It is given by the relation:

$$\text{ctg}\ \delta=-\frac{b_i}{b_e}\frac{D_e\beta_0^2}{(1+y)\omega}(QS-\phi). \qquad (23)$$

Inserting (22) and (23) into (19) we find the relation between S and ϕ:

$$S=\phi\frac{1{\cdot}3+X^2-0{\cdot}7\phi}{1{\cdot}5+X^2-0{\cdot}9\phi} \qquad (24)$$

By specifying a definite value for ϕ and substituting (24) into (21) we obtain an equation for X and y. Stationary oscillations are possible only where equation (21) is satisfied at the unique point where the curves defined by the right and left-hand sides of (21) touch tangentially. From this condition we can determine the theoretical curve $v^* = v^*(y, \phi)$. On the other hand we know the form of the dependence $v^*(E/p)$ experimentally and, furthermore, according to (19)

$$E/p=f\left(\sqrt{\frac{1+y}{\phi}}\,ap\right),$$

where $f(ap)$ is the experimentally determined dependence of E/p on ap in the absence of a magnetic field.

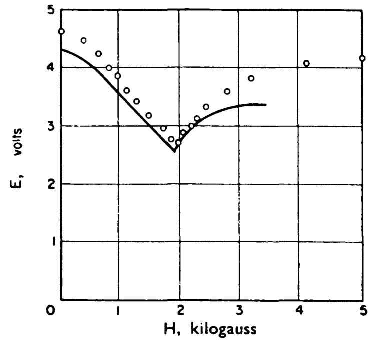

FIG. 1.

We will find $y = y(\phi)$, and consequently the relation between E and H in the presence of steady oscillations, by finding the intersection of the curve $v^* = v^*(y)$ calculated in this way with the curve defined by the condition $\text{Im}\ \omega = 0$.

In Fig. 1 the curve obtained on the basis of these calculations is compared with the experimental results of HOH and LEHNERT for the values $a = 1$ and $p = 0{\cdot}89$. It can be seen that there is not only qualitative agreement with the experimental points but, furthermore, good quantitative agreement up to a field $H = 3$ kgauss. At this value of the field $\phi = 4$, i.e. the averaged flux of particles at the wall is four times greater than the diffused flux. The departure of the theoretical curve from the experimental points when H is increased further may be due either to the approximations made in solving the equations or to the appearance of oscillations having a shorter wavelength (in particular, those with $m > 1$).

4. SCALE RELATIONS

Equations (5) and (6) describing the diffusion processes in the plasma have the specific property that length, time and magnetic field enter into them only in combination with the pressure; that is, in the form ap, tp, H/p. This is because Z is proportional to, and b, D and τ inversely proportional to the pressure. Equations (5) and (6) still depend implicitly on the electron and ion temperatures and these are determined by the energy balance, i.e. ultimately by the quantity E/p.

This means that if we have two geometrically similar systems with identical parameters ap, E/p and H/p equations (5) and (6) coincide for both systems and the processes which take place in these systems will differ only by the time scale. This is still the case when the action of the magnetic field on the ions and the diffusion and inertia of the ions are taken into account.

Furthermore, equations (5) and (6) are linear with respect to the density n, i.e. the absolute magnitude of n or, in other words, the magnitude of the electric current in no way affects the processes which take place in the column.

Thus, if the degree of ionization is so small that electron-ion collisions can be neglected, then we have the following scale relation: E/p and ω/p are functions of ap and H/p only and are independent of the magnitude of the current. A comparison of the experimental dependence of E on H obtained by HOH and LEHNERT for different values of a and p shows that such a scale relation is in fact valid. This can also be seen from the table of critical field values H_c (Table 1).

5. CONCLUSION

We have seen that the positive column of a gas discharge becomes unstable if the magnetic field exceeds a certain critical value H_c. If the magnetic field is only a little greater than H_c then regular oscillations build up in the plasma with a frequency

$$f = \frac{\omega}{2\pi} \sim \frac{10 b_i D_e}{\Omega \tau b_e a^2}$$

which, for helium and the experimental conditions of LEHNERT and HOH, is roughly equal to 10 kc/sec. It seems possible that such oscillations would explain the effect observed by. FABRIKANT and ROKHLIN (1938) who found that the maximum electron density occurred off the axis of the discharge when a magnetic field was applied.

One might suppose that with a further increase of H even higher harmonics will be excited and the oscillations of the plasma channel will eventually become irregular and chaotic. Such a situation can be identified with the turbulent state of the plasma first mentioned by BOHM (GUTHRIE and WAKERLING, 1949). By analogy with normal turbulence, one might attempt to formulate a theory of such a turbulent discharge but to do this it would be desirable to have more complete experimental data on the intensity and spectrum of the oscillations.

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Stability of Plasmas Confined by Magnetic Fields[1]

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In this paper, we examine the question of the stability of plasmas confined by magnetic fields. Whereas previous studies of this problem have started from the magnetohydrodynamic equations, we pay closer attention to the motions of individual particles. Our results are similar to, but more general than, those which follow from the magnetohydrodynamic equations.

I. INTRODUCTION

The problem of the behavior of highly ionized plasmas in electromagnetic fields has recently become the object of considerable interest (*1*). Although there is little more involved in the problem than Newton's laws and Maxwell's equations, there are many questions one can ask to which the answers have been by no means obvious or even easily calculable. Two of the several rather broad areas into which these questions fall are the following.

(a) The existence and properties of stationary solutions of the equations. Here, "stationary" is not meant to imply that fields and particle positions or velocities are absolutely constant, but that averages of these quantities over times longer than the Larmor period and over the statistical particle distribution are constant. Collisions between the particles are to be ignored. Effects to be considered are the diamagnetic and electric effects of the charged particles on the fields, and, conversely, the effects of the fields in influencing the particle distribution function.

(b) The stability of these stationary solutions under arbitrary perturbations of the plasma configuration. Here again collisions are to be ignored. It is known that collisions produce a diffusion of charged particles across magnetic fields, but we are interested here in instabilities, similar to those in hydrodynamics, in which locally coordinated motions of the plasma occur under the influence of the average electromagnetic fields.

Although problems falling under category (a) have been solved in only the

[1] Work performed under the auspices of the Atomic Energy Commission.

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simplest cases, there seems to be no question of the existence and general nature of the solutions. We turn our attention in this paper to the problem of stability.

One method of attacking the problem is through the Boltzmann equation (actually the Liouville equation, since collision terms are dropped). On taking the first three moments of the Boltzmann equation with respect to the velocity, one obtains macroscopic equations expressing the conservation of mass, momentum, and energy (*2*). These equations cannot be solved, since they contain more moments of the velocity distribution as unknowns than there are equations. The usual procedure at this point is to assume that the pressure tensor is a simple scalar, and remains so during the perturbations, and to neglect several higher moments. One then arrives at one or another of the forms of the magnetohydrodynamic equations.

This procedure is risky because of the guessing involved. It would appear to be especially risky in a problem such as the present one in which there are no collisions helping to keep the pressure tensor nearly a scalar, although it can be argued that the restraining magnetic field partly plays the role of collisions. To put the objection as strongly as possible, consider the following example. For an ordinary gas, if one assumes that the pressure tensor is a scalar, the above procedure leads to the equations of hydrodynamics, even if the particles are assumed not to undergo collisions. Characteristic of the solutions of these equations are the familiar sound waves. But certainly in a gas without collisions, disturbances will not at all propagate and oscillate like sound waves.

While this illustration of the danger is undoubtedly overdrawn, it seems wise at least to check stability results based on the magnetohydrodynamic equations (*3*) by other methods when possible. In this paper, we approach the stability problem by discussing the orbits of individual particles.

Watson and Brueckner (*4*) have also succeeded in improving on the magnetohydrodynamic approximation by a more careful treatment of the Boltzmann equation (which, of course, knows all the answers). While this formal approach through the Boltzmann equation may likely provide the most accurate and rigorous treatment of more complex situations, we believe the more mechanistic approach used in this paper better illustrates the physical factors involved in the instability.

II. FIRST ORDER ORBIT THEORY

In this section, we review briefly the theory of charged particle orbits in slowly varying fields. This theory has been developed by Alfvén (*1*), Spitzer (*5*), and others.

In a magnetic field **B** which is constant in space and time, the trajectory of a charged particle is a helix with axis parallel to the field lines. The center of revolution of the particle is called its "guiding center." The guiding center moves with constant velocity along a field line. The kinetic energies due to motion parallel to **B**, denoted by $w_{\parallel}$, and perpendicular to **B**, denoted by $w_{\perp}$, are arbitrary. The radius r of the helix and the angular (Larmor) frequency are[2]

$$r = \frac{mv_{\perp}c}{eB}, \qquad \omega = \frac{eB}{mc} \tag{1}$$

where m and e are the mass and charge of the particle, $v_{\perp}$ is the magnitude of the component of velocity perpendicular to B, and c is the light velocity. The circular motion of the particle relative to its guiding center produces, after a time average, a magnetic moment μ which is antiparallel to **B** with magnitude

$$\mu = w_{\perp}/B. \tag{2}$$

Thus, plasmas tend to be diamagnetic. However, we shall not introduce a magnetic susceptibility to cover these diamagnetic effects, but shall use directly the average electrical currents which are responsible for the diamagnetism. If one has a distribution of magnetic dipoles giving rise to a magnetic moment **M** per unit volume, the electric current $\mathbf{J}_m$ associated with **M** (which can be considered to be the source of **M**) is given by the expression

$$\mathbf{j}_m = \nabla \times \mathbf{M} \tag{3}$$

familiar in electromagnetic theory. In our case

$$\mathbf{M} = -\frac{W_{\perp}}{B^2}\mathbf{B} \tag{4}$$

where $W_{\perp}$ is the total "perpendicular" kinetic energy per unit volume.

The effect of a constant electric field **E** in addition to the magnetic field is well known. The component of **E** parallel to **B** simply accelerates the guiding center in that direction. The component of **E** perpendicular to **B** can be transformed away (provided $E_{\perp} < B$) by going to a coordinate system moving with velocity

$$\mathbf{v}_E = \frac{c}{B^2}\mathbf{E} \times \mathbf{B}. \tag{5}$$

Since in this system the perpendicular motion is simply circular, we see that, in the original system, the guiding center drifts across the field lines with the velocity $\mathbf{v}_E$ given by Eq. (5).

Since, according to Eq. (5), charges of both signs undergo the same drift, no electrical current results from $\mathbf{E}_{\perp}$ if the plasma is neutral. This result is not altered by collisions between the charged particles, as long as there are no neutrals. One is tempted to say that, in a completely ionized plasma in a magnetic field, the electrical conductivity is equal to zero. On the other hand, one can

[2] Gaussian units are used throughout this paper.

have currents in a plasma [for example, $\mathbf{j}_m$ given by Eq. (3)] without inducing potential drops, and this is the basis for the common statement that the conductivity of an ionized plasma is infinite if collisions are neglected. The explanation of this paradox is that electric fields and currents are not connected by any relation as simple as that defining a conductivity.

If the particles are subject to an external force $\mathbf{F}$ (such as gravity), currents can result. Such a force produces the same drifts as an electric field $\mathbf{E} = \mathbf{F}/e$, which are in opposite directions for charges of opposite sign.

While there is no electric current proportional to $\mathbf{E}_\perp$, there is a current proportional to $\partial\mathbf{E}_\perp/\partial t$. Thus we may say that the plasma contributes a dielectric constant. To see this, let us calculate the displacement of the guiding center of a particle when $\mathbf{E}$ is suddenly changed by $\Delta\mathbf{E}_\perp$. Let us view the particle, both before and after the change, in the coordinate systems in which its guiding center is at rest. The difference in velocity of these two systems is, according to (5),

$$\Delta\mathbf{v} = \frac{c}{B^2}\Delta\mathbf{E}_\perp \times \mathbf{B}$$

Now if the particle is at the point $\mathbf{R}$ and has velocity $\mathbf{v}$, its guiding center is at the point [from (1)]

$$\mathbf{r} = \mathbf{R} + \frac{mc}{eB^2}\mathbf{v} \times \mathbf{B}.$$

Therefore, the instantaneous displacement of the guiding center is

$$\Delta\mathbf{r} = \frac{mc}{eB^2}(-\Delta\mathbf{v}) \times \mathbf{B} = \frac{mc^2}{eB^2}\Delta\mathbf{E}_\perp .$$

This step is in opposite directions for charges of opposite sign, but in a neutral plasma the ions contribute by far most of the current. One takes account of this effect in first order orbit theory either by including a polarization current (emu)

$$\mathbf{j}_P = \frac{\rho c}{B^2}\frac{\partial\mathbf{E}_\perp}{\partial t} \qquad (6)$$

where ρ is the mass density, or by introducing a dielectric constant

$$\epsilon = 1 + \frac{4\pi\rho c^2}{B^2}. \qquad (7)$$

Either of these expressions is clearly valid only in times long compared to the Larmor periods. An alternative derivation of (7) can be had by taking the low-frequency limit of the classical expression for the dielectric constant in the

Faraday effect. Because the dielectric constant (7) is often large, its effect must be included in the first order orbit theory.

Let us now examine the effects of derivative of **B**. If **B** has a time derivative, electric fields are generated. One effect of these electric fields is to make the guiding centers move in a way described by Eqs. (5), (6), and (7) above. In this connection, it is sometimes said that the result of the drift (5) is to make guiding centers move as if they were attached to the magnetic field lines. This statement is true only when $\nabla \times \mathbf{E}_{\parallel}$ is parallel to **B**. We shall not prove this, since the statement has no essential utility even when true. An example where the statement is not true is the following. On a field produced by a current in a straight wire, superimpose another magnetic field parallel to the wire. The field lines are then helices. Now vary the strength of the component parallel to the wire with time. In this case, the field lines cannot be considered to move with the velocity (5).

Another effect of the electric field induced by a changing **B** is to change the energy $w_{\perp}$ of the particle. It is well known, and easily shown, that, if the magnetic field changes only slightly per Larmor cycle, the ratio $w_{\perp}/B$ is almost constant. Thus, according to (2), the magnetic moment (and angular momentum) of the particle is an adiabatic invariant. The same result holds if the magnetic field is static and the particle moves from a region of one field strength to another, as can be seen by simply transforming the guiding center to rest.

To see the effects of spatial derivatives of **B**, we divide the nine terms of the tensor $\nabla\mathbf{B}$ into groups of terms and consider each group separately. For convenience, we choose a cartesian coordinate system with the z-axis tangent at the origin to the center line of a bundle of field lines. In zeroth approximation (if there were no derivatives), a guiding center once on this line would stay on it and move with constant velocity. We shall find the effects proportional to the first power of the first derivatives of **B**.

Consider first the diagonal terms of the tensor $\nabla\mathbf{B}$. If $\partial B_x/\partial x$ and $\partial B_y/\partial y$ do not vanish, the bundle of lines has an angular divergence, as shown in Fig. 1. A particle with Larmor orbit as shown then sees a component of force parallel to the central line and in the direction of decreasing field strength. This force is

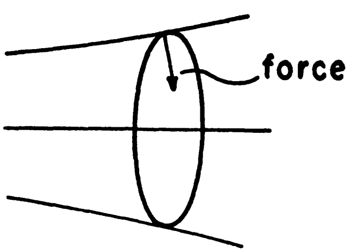

FIG. 1. Effect of angular divergence of field lines.

easily found (using $\nabla \cdot \mathbf{B} = 0$) to be

$$F_{\|} = -\frac{w_{\perp}}{B} \frac{\partial \, |B_z|}{\partial_z} = (\boldsymbol{\mu} \cdot \nabla) B. \tag{8}$$

Since $\boldsymbol{\mu}$ is antiparallel to **B** and constant in magnitude as the guiding center moves along the central line, the effect of this force on the motion parallel to **B** can be viewed as resulting from a potential energy $\mu B = w_{\perp}$. One thus finds that $w_{\|} + \mu B = w_{\|} + w_{\perp}$ is a constant of motion. This result is, of course, exactly true in a static magnetic field; that our derivation yields this result should be regarded as verifying the model on which the derivation is based rather than the result. The model shows how particles are reflected from a region where the field strength increases sufficiently to bring $w_{\|}$ to zero.

Consider next the terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$. If these two terms do not vanish, the central line of the bundle is curved (Fig. 2), and the radius of curvature is, in general, given by

$$\frac{\mathbf{R}}{R^2} = -(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_0$$

where $\mathbf{B}_0$ is a unit vector field parallel to **B**. If, in zeroth approximation, the guiding center is assumed to move along the central line, the particle will experience a centrifugal force

$$\mathbf{F} = \frac{m v_{\|}^2}{R^2} \mathbf{R}$$

which is perpendicular to **B**, and this will result in a drift of the guiding center with velocity

$$\mathbf{v}_1 = \frac{c}{eB^2} \mathbf{F} \times \mathbf{B} = \frac{2c}{e} \frac{w_{\|}}{B} \mathbf{B}_0 \times (\mathbf{B}_0 \cdot \Delta)\mathbf{B}_0. \tag{9}$$

This drift, which is in opposite directions for positive and negative charges,

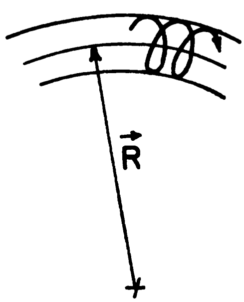

FIG. 2. Effect of curvature of field lines.

makes an electric current which we shall call

$$\mathbf{j}_1 = \frac{Ne\mathbf{v}_1}{c} \tag{10}$$

where N is the density of particles.

Next take the terms $\partial B_z/\partial x$ and $\partial B_z/\partial y$. If these terms do not vanish, the magnitude B varies in a direction perpendicular to $\mathbf{B}$. This gradient causes a force on the magnetic moment of the particle

$$\mathbf{F} = -\mu \nabla B$$

which is perpendicular to $\mathbf{B}$, and this results in a drift (Fig. 3) of the guiding center with velocity

$$\mathbf{v}_2 = \frac{c}{e} \frac{w_\perp}{B^2} \mathbf{B}_0 \times \nabla B. \tag{11}$$

This drift again makes an electric current, which we shall call

$$\mathbf{j}_2 = \frac{Ne\mathbf{v}_2}{c}. \tag{12}$$

The two remaining terms of the tensor ∇B are $\partial B_x/\partial y$ and $\partial B_y/\partial x$. When these two terms are nonvanishing, the lines of the bundle twist or shear about the central line. These terms slightly change the shape of the Larmor orbit, but produce no drifts in first order.

The general problem is now to solve Maxwell's equations with the currents $\mathbf{j}_m$, $\mathbf{j}_P$, $\mathbf{j}_1$, and $\mathbf{j}_2$, and to carry forward the evolution of the particle distribution by means of the drifts in the perpendicular directions and by Eq. (8) in the parallel direction.

III. MECHANISM OF INSTABILITY IN THE PARTICLE PICTURE

We now assume that a stationary solution of the preceding equations and Maxwell's equations has been found, and prepare to discuss the stability of the solution under an arbitrary small perturbation. In this section, we show, in a simple case, the details of the mechanism of instability. In more complicated

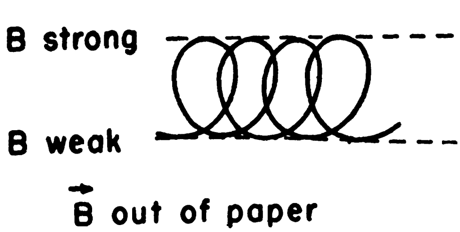

FIG. 3. Effect of a gradient of B perpendicular to $\mathbf{B}$.

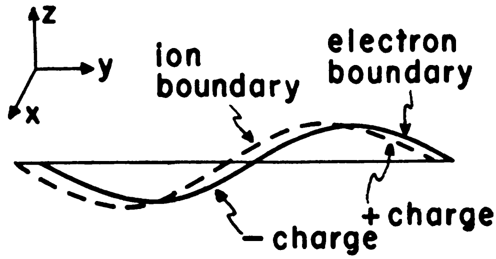

FIG. 4. Charge separation in gravitational instability.

problems, it is difficult to follow the detailed mechanism. In such cases, the more powerful energy principle developed in Sections IV and V may be used to decide which perturbations, if any, are energetically permitted.

The instability of a plasma supported under gravity by a magnetic field has been studied by Kruskal and Schwarzschild,[3] who used the one-fluid equations of magnetohydrodynamics. We consider now the same problem from the point of view of first-order orbit theory.

Let the gravitational acceleration **g** be in the $-z$ direction, and the magnetic field **B** lie in the x direction. Assume that the plasma initially has constant temperature and density, and has a sharp boundary on the lower (z) side. Let this boundary be rippled by a perturbation

$$\Delta z = a \sin ky. \tag{13}$$

Thus the B lines are parallel to the surface. We suppose that initially (13) gives the surfaces of both ions and electrons. This situation is soon altered. The drift due to the gravitational force

$$\mathbf{v}_g = c \frac{m}{e} \frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{14}$$

will make the ions drift in the $-y$ direction, the electrons in the $+y$ direction. This will cause charge separation in a layer on the surface of the plasma, as indicated in Fig. 4. The electron drift is much smaller than the ion drift, and will be neglected.

If the amplitude a of the perturbation is small compared to the wavelength, the time rate of change of surface charge density $\sigma(y)$ is

$$\begin{aligned} \frac{\partial \sigma}{\partial t} &= -Ne \mid v_g \mid ka \cos ky \\ &= -\frac{NMcg}{B} ka \cos ky \end{aligned} \tag{15}$$

[3] A brief sketch of the essentials of the mechanism presented here is given in Kruskal's and Schwarzschild's paper (*3*).

where N is the ion density and M the ion mass. In considering the surface charge density, we are assuming that the charged layer is quite thin. For small amplitude a, we may also assume that this charged layer lies in the original plane of the boundary. The electric field of the charged layer is then easy to compute.

Remembering that the plasma has a dielectric constant

$$\epsilon \approx \frac{4\pi NMc^2}{B^2} \tag{16}$$

which we shall assume is large compared to unity, we find that a surface charge density

$$\sigma = \sigma_0 \cos ky \tag{17}$$

produces the electric fields, in the plasma,

$$E_y = \frac{4\pi\sigma_0}{\epsilon} \sin kye^{-kz}$$

$$E_z = \frac{4\pi\sigma_0}{\epsilon} \cos kye^{-kz}$$

These electric fields also cause a drift of the plasma, according to Eq. (5). The components of this drift velocity are

$$v_y = \frac{4\pi c\sigma_0}{\epsilon B} \cos kye^{kz}$$

$$v_z = \frac{4\pi c\sigma_0}{\epsilon B} \sin kye^{-kz}$$

This velocity field is divergenceless, and, therefore, does not change the density of the plasma, except at the boundary. The velocity v_z at the boundary ($z \simeq 0$) causes the amplitude of the perturbation to grow according to

$$\frac{da}{dt} = -\frac{4\pi c}{\epsilon B}\sigma_0 . \tag{18}$$

Comparing Eqs. (15) and (17), we find

$$\frac{d\sigma_0}{dt} = -\frac{NMcgk}{B} a. \tag{19}$$

Finally, combining (16), (18), and (19), we find

$$\frac{d^2a}{dt^2} = gka$$

with solutions

$$a(t) = a_0 \exp (\pm \sqrt{gk}t) \tag{20}$$

If the sign of g is reversed, the solutions are oscillatory (stable).

It is interesting to note that the rate of growth here is exactly the same as in the Taylor instability of a fluid supported under gravity by a second fluid which is weightless. Thus the charge separation is able to overcome exactly the restraining influence of the magnetic field. This exact compensation occurs, however, only in the limit $\epsilon \gg 1$. If ϵ is not large, ϵ has to be replaced in (18) by $1 + \epsilon = 2 + 4\pi NMc^2/B^2$; the rate of growth is then reduced and is eventually proportional to $\sqrt{N}$ for low-density plasmas.

The essential mechanism of the instability is the charge separation produced by the gravitational force. Any force perpendicular to **B** which is independent of the sign of the charge will cause such a charge separation. In Section II, we saw two such forces, namely:

(1) the centrifugal force on particles moving along curved B lines, for which $\mathbf{g}$ in the above analysis can be replaced by $\mathbf{R}(v_\parallel^2/R^2)$, where $\mathbf{R}$ is again the radius of curvature of the line; and

(2) the force due to a gradient of B in a direction perpendicular to $\mathbf{B}$, leading to the drift (11), for which $\mathbf{g}$ can be replaced by $-(v_\perp^2/2)(\nabla_\perp B/B)$.

If $\nabla \times \mathbf{B} \approx 0$ (i.e., no appreciable currents in the region being considered), then

$$\frac{1}{B} \nabla_\perp B \approx -\frac{\mathbf{R}}{R^2}$$

and, to include the two effects, g has to be replaced by

$$\mathbf{g} \rightarrow \frac{\mathbf{R}}{R^2}\left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right) \tag{21}$$

If the vector on the right of (21) points away from the plasma, i.e., if the plasma boundary is convex along field lines, the boundary is unstable. When particles wander back and forth along field lines, passing through regions of varying curvature, an appropriate average of the expression (21) has to be taken. The type of average depends on the field geometry, and will be illustrated in Section V.

IV. SIMPLE THERMODYNAMIC DESCRIPTION OF INSTABILITY

In this section and the next, we discuss the instability from the thermodynamic point of view that, if a state of lower potential energy is available to the plasma, it will seek it out, the extra energy going into kinetic energy of the instability. This thermodynamic approach rests on the assumption that the equations of motion will always allow the plasma to move in the direction of lower potential energy. In Section III we have seen in detail, in a simple problem, how such an energy lowering motion is achieved by the plasma. We henceforth assume that the necessary motions are always possible.

If one takes as the starting point the magnetohydrodynamic equations, it is easy to justify the energy principle, since the equations of motion can be shown to follow from minimization of the potential energy, as in mechanics. No such general proof exists, to the authors' knowledge, for the more nearly correct treatments in which the particle pressure tensor is assumed (realistically) to be not a simple scalar. However, one may certainly say that if the existing configuration has a lower energy than all neighboring configurations, the system must be stable. It is conceivable that "overstable" solutions exist with rapid oscillations, so that the concept of adiabatic invariants becomes invalid. This seems unlikely intuitively, but would invalidate Section V.

The potential energy is the sum of the magnetic field energy $B^2/8\pi$ and the internal energy of the plasma. For simplicity, we make in this section the same assumption as is made in magnetohydrodynamics, namely, that the pressure tensor is a scalar. (This restriction will be removed in Section V.) It then follows that the internal energy of the plasma per unit mass is

$$E_p = \frac{pv}{\gamma - 1} \tag{22}$$

where p is the pressure and v the specific volume. In any adiabatic motion,

$$p \sim v^{-\gamma} \tag{23}$$

The simplest situation displaying instability is the gravitational problem dis-

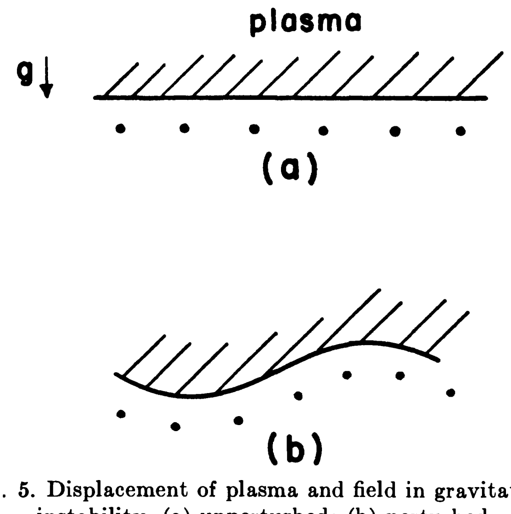

FIG. 5. Displacement of plasma and field in gravitational instability. (a) unperturbed; (b) perturbed.

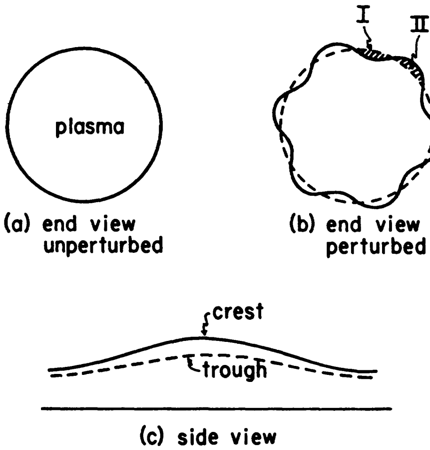

FIG. 6. Illustration of flute-type instability.

cussed in Section III. In Fig. 5b, we see a possible perturbation wherein some of the regions occupied by plasma and magnetic field have been interchanged. We may take the perturbation to be such that the magnetic field and plasma internal energy are unchanged so the only change in potential energy of the system is the change in gravitational potential due to lowering of the plasma. Hence, the system is unstable. We can see that an identical discussion would apply to the case of Taylor instability in which the plasma and field are to be replaced by heavy and light fluids, respectively.

A more interesting case is that in which the plasma is confined by a curving magnetic field. For simplicity, we will discuss an axially symmetric case, periodic in the z direction in which the plasma is confined near the axis. We will also discuss the limit where the plasma pressure is small compared to the magnetic field pressure. In this limit, the magnetic field is nearly identical to the vacuum magnetic field so that any distortion of the field increases its energy. Hence, the only dangerous perturbations are those which leave the field unchanged. Such deformations are the flutes shown in Fig. 6.

The flutes are thus constructed so their surface is bounded by lines of the unperturbed magnetic field. It is easily seen that the net effect of the flutes is to interchange the field and matter that were originally contained in the flux tube I with those contained in flux tube II, and we will now proceed to calculate the energy change produced by such an interchange.

Now the magnetic energy in a flux tube is

$$E_M = \int \frac{B^2}{8\pi} dV = \int \frac{B^2}{8\pi} dlA \tag{24}$$

where l is the length along the tube, A is the cross sectional area of the tube, and

$\int dl$ is the integral along a complete flux line. But

$$BA = \phi = \text{Flux} \tag{25}$$

which is constant along a flux tube. So

$$E_M = \frac{1}{8\pi}\phi^2 \int \frac{dl}{A}. \tag{26}$$

Thus the change in magnetic energy on interchange of flux tubes I and II is:

$$\Delta E_M = \frac{1}{8\pi}\left[\left\{\phi_{\mathrm{I}}^2 \int_{\mathrm{II}} \frac{dl}{A} + \phi_{\mathrm{II}}^2 \int_{\mathrm{I}} \frac{dl}{A}\right\} - \left\{\phi_{\mathrm{I}}^2 \int_{\mathrm{I}} \frac{dl}{A} + \phi_{\mathrm{II}}^2 \int_{\mathrm{II}} \frac{dl}{A}\right\}\right] \tag{27}$$

Thus, if we want to require no change in magnetic energy, we must interchange tubes containing equal amounts of flux, i.e., $\phi_{\mathrm{I}} = \phi_{\mathrm{II}}$. (The magnetic field is then left unchanged.)

To calculate the change in material energy, we use Eqs. (22) and (23).

The volume V of a flux tube is given by

$$V = \int dl\, A = \phi \int \frac{dl}{B}. \tag{28}$$

Hence, the change in material energy is given by

$$\Delta E_p = \frac{1}{\gamma - 1}\left\{p_{\mathrm{I}} \frac{V_{\mathrm{I}}^{\gamma}}{V_{\mathrm{II}}^{\gamma}} V_{\mathrm{II}} + p_{\mathrm{II}} \frac{V_{\mathrm{II}}^{\gamma}}{V_{\mathrm{I}}^{\gamma}} V_{\mathrm{I}} - p_{\mathrm{I}} V_{\mathrm{I}} - p_{\mathrm{II}} V_{\mathrm{II}}\right\}. \tag{29}$$

We have used the scalar pressure result that p is constant along a line. If the flux tubes are nearby, we may expand

$$\left.\begin{aligned} p_{\mathrm{II}} &= p_{\mathrm{I}} + \delta p \\ V_{\mathrm{II}} &= V_{\mathrm{I}} + \delta V \end{aligned}\right\} \tag{30}$$

and find

$$\Delta E_p = \delta p \delta V + \gamma p \frac{(\delta V)^2}{V} = V^{-\gamma} \delta(p V^{\gamma}) \delta V. \tag{31}$$

The condition for stability is that $\Delta E > 0$. In general, p will decrease as we move outwards from the axis and as we approach the edge of the plasma, it must go to zero. Hence, in this region $\delta(pV^{\gamma})$ is negative since as $p \rightarrow 0$ $|\delta p/p| > |\gamma(\delta V/V)|$ and our condition for stability becomes

$$\delta \int \frac{dl}{B} < 0. \tag{32}$$

Equation (32) is a general result, dependent only on the use of the magnetohydrodynamic equations and the assumption of low plasma pressure.

We now discuss the geometrical significance of Eq. (32).

Figure 7 shows two nearby lines of flux. D is the perpendicular distance between them at any point. Since $\mathbf{B}$ is curl free, $\int B\,dl$, the magnetic potential is the same for both lines at the points joined by D. Therefore,

$$\delta \int \frac{dl}{B} = \int_{\text{II}} \frac{dl}{B} - \int_{\text{I}} \frac{dl}{B} = \int B\,dl \left[\frac{1}{B_{\text{II}}^2} - \frac{1}{B_{\text{I}}^2}\right] = B\,dl\delta\left(\frac{1}{B^2}\right) \tag{33}$$

From the same fact, it follows (integrate around the dotted rectangle) that

$$\delta B/B = D/R \tag{34}$$

where R is the radius of curvature of the line, positive if the center of curvature lies outside the plasma. Therefore,

$$B\delta(1/B^2) = -2(D/BR)$$

Here D can be eliminated by using the fact that the flux between the lines is constant along their length

$$2\pi r D B = \phi = \text{constant} \tag{35}$$

where r is the radius of the line relative to the axis of symmetry. Thus the condition for stability (32) in the magnetohydrodynamic approximation becomes

$$\int \frac{dl}{RrB^2} > 0. \tag{36}$$

Note that, by its definition, R is positive at the ends and negative in the middle, so the middle region tends to make things unstable. In general, the fact that B is large at the ends will make the middle region dominate so that the system appears unstable. We may note that $dl/R \approx d\theta$, the angle of the flux line with the z-axis, and that Br^2 is roughly a constant so that (36) may be approximated as

$$\int \frac{d\theta}{B^{3/2}}.$$

Since B is smaller in the region of negative $d\theta$, the situation is unstable.

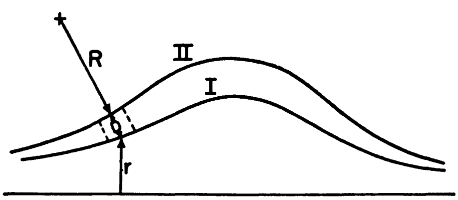

FIG. 7. Illustration of quantities involved in flute instability.

V. CORRECT TREATMENT OF PLASMA ENERGY

In this section, we shall discuss the flute-type instability of Section IV, trying, however, to calculate the change in internal energy of the plasma correctly rather than by the magnetohydrodynamic approximation. To do this, we first look at individual particles, calculating the change in energy of a particle as the line to which it is tied moves in a flute-type instability. First we note that there are two adiabatic invariants of the motion

$$w_{\perp}/B = \mu \tag{37}$$

$$\int v_{\|} \, dl = J. \tag{38}$$

Here $w_{\perp}$ is the particle energy in the plane transverse to the field, and $v_{\|}$ is its velocity along the line, the integral being taken between turning points or over a period of the motion.

Equation (37), the constancy of the magnetic moment, has been discussed in Section II. It is related to the conservation of angular momentum about the direction of the line and is valid so long as the Larmor radius and period are small compared to characteristic lengths and periods of the instability.

Equation (38), the invariance of the longitudinal action, is related to conservation of linear momentum along the line. It is valid if the period of the particle in its orbit is small compared to the instability period. This is certainly true at the borderline of stability. We shall not prove Eq. (38) here, relying on the obvious analogy with classical mechanics. It has been verified in a lengthy derivation by Chew, Low, and Goldberger.

We also assume that particles remain tied to field lines so that, in particular, all particles which were orbiting on a field line before a deformation are orbiting on the same field line following the deformation. This is true in the present example, illustrated by Fig. 6, so long as the Larmor radius is small compared to the wavelength of the deformation and the Larmor frequency is large compared to the characteristic frequency of the deformation.

Since the orbit of a particle, except for uninteresting phases, is determined by its field line and by the two constants of the motion μ and J, it is clear that a prescription of the deformation which describes the new configuration of the field lines is adequate to define completely the new state of the system. Thus the constraints are holonomic, and we can indeed use the variational principle to determine stability.

It is now a straightforward procedure to construct the change in energy of the system for an arbitrary displacement. The algebra, however, is quite complex, and this has not yet been done.

We shall, however, solve the problem for the same case treated in Section IV, the flute displacement which interchanges field lines. Again it is sufficient, at least near the edge of the plasma, to inquire whether the energy of the particles on a field line increases or decreases when that line is interchanged with a line further out. In the low plasma density case, **B** is curl free, and we may incorporate many of the results of Section IV.

We now calculate the change in energy for a given particle as its field line is displaced from I to II. Using (37) and (38), we may write

$$\int \sqrt{w_{II} - \mu B_{II}}\, dl = \int \sqrt{w_I - \mu B_I}\, dl \tag{39}$$

where w is the total kinetic energy of the particle. Again, since it is convenient to compare B_I and B_{II} at points connected by a perpendicular to the field lines, we introduce the magnetic potential $\int B\, dl$ and write

$$\int \frac{\sqrt{w_{II} - \mu B_{II}}}{B_{II}}\, d\left(\int B_{II}\, dl_{II}\right) = \int \frac{\sqrt{w_I - B_I}}{BI}\, d\left(\int B_I\, dl_I\right)$$

We now expand

$$\left.\begin{aligned} w_{II} &= w_I + \delta w \\ B_{II} &= B_I + \delta B \end{aligned}\right\} \tag{40}$$

and obtain

$$\int \sqrt{w - \mu B}\left(1 + \frac{1}{2}\,\frac{\delta w - \mu \delta B}{w - \mu B}\right)\left(1 - \frac{\delta B}{B}\right) dl = \int \sqrt{w - \mu B}\, dl. \tag{41}$$

We note that the fact that the turning points of orbits I and II may differ does not affect the value of the action integral since the integrand vanishes at the turning point.

We can now solve Eq. (41) for δw (a constant)

$$\delta w = \frac{\displaystyle\int \frac{\delta B(2w/B - \mu)}{\sqrt{w - \mu B}}\, dl}{\displaystyle\int \frac{dl}{\sqrt{w - \mu B}}} \tag{42}$$

We use the result of Eq. (34) and (35),

$$\delta B = C\,\frac{1}{Rr} \tag{43}$$

where C is an irrelevant constant.

So, dropping constants, we find

$$\delta w \sim \frac{\displaystyle\int \frac{2w - \mu B}{RrB\sqrt{w - \mu B}}\, dl}{\displaystyle\int \frac{dl}{\sqrt{w - \mu B}}} \tag{44}$$

This result is quite similar to the magnetohydrodynamic one. As in the latter case, the regions near the field maximum contribute to stability. In the present formula, the effect of this stable region is somewhat enhanced due to the square root in the denominator which becomes small near the turning point. Thus, in most geometries, particles with turning points near the field maximum will have stable orbits. In all cases, of course, particles which are reflected near the center where R is still negative will have unstable orbits. For most conceivable distributions, the effect of these will predominate, and the whole system is unstable. One can note that the most energetically favorable situation is for the "unstable" orbits to be concentrated in the crest of the flute, and the "stable" orbits in the trough. Thus one might suppose that a sort of distillation occurs until only stable orbits remain. So far, an adequate mechanism for this has not been found.

We now sum the above equation over all particle orbits tied to the line in order to find the total energy change of the plasma

$$\Delta E_P = \int \delta w(\mu, w) N(\mu, w)\, d\mu\, dw. \tag{45}$$

We shall try to evaluate the integral in this expression in more familiar terms.

First we note that the density $\rho(w, \mu)$ of particles of a given kind (specified w and μ) at a position l along the line is proportional to:

1. The number of such particles present per line $N(w, \mu)$
2. The fraction of its time which it spends between l and $l + dl$

$$\frac{\dfrac{dl}{v_{\|}}}{\displaystyle\int \frac{dl}{v_{\|}}} = \frac{\dfrac{dl}{\sqrt{w - \mu B}}}{\displaystyle\int \frac{dl}{\sqrt{w - \mu B}}}$$

3. The density B of flux lines at l.

If, in addition, we use the fact $2w - \mu B = 2w_{\|} + w_{\perp}$, we can rewrite Eq. (45) in the form

$$\Delta E_P \sim \int dl \int d\mu\, dw\, \frac{\rho(\mu, w)[2w_{\|} + w_{\perp}]}{RrB^2} \tag{46}$$

Note that, in this expression, the parallel and perpendicular kinetic energies enter with just the ratio already seen in Eq. (21), Section III. We now see what sort of average has to be taken of the two forces which drive the instability: the centrifugal force $2w_{\parallel}/R$, and the force $w_{\perp}/R$ on the magnetic moment due to the gradient of B in the perpendicular direction. The type of average in Eq. (46) is such as to decide whether the net drift in angle about the axis of the system is positive or negative. It may be seen independently that this is the correct average to take.

We may also express the result (46) in terms of the pressure tensor

$$\left.\begin{aligned} p_{\parallel} &= \int d\mu\, dw\, \rho(\mu, w) w_{\parallel} \\ p_{\perp} &= (\tfrac{1}{2}) \int d\mu\, dw\, \rho(\mu, w) w_{\perp} \end{aligned}\right\} \tag{47}$$

with these definitions, we may finally write

$$\Delta E_P \sim \int dl\, \frac{p_{\parallel} + p_{\perp}}{RrB^2} \tag{48}$$

and our stability criterion becomes

$$\int dl\, \frac{p_{\parallel} + p_{\perp}}{RrB^2} > 0 \tag{49}$$

It will be observed that, if we make the approximation of magnetohydrodynamics that $p_{\parallel} = p_{\perp} =$ constant along a line, this reduces to the result of the previous section, Eq. (36). Thus (49), in general, again appears to predict instability, although it is often possible to construct certain orbit distributions which are stable.

VI. NONLINEAR EFFECTS ON THE FLUTE GROWTH

In this section, we examine two nonlinear effects on the growth of the flutes. The first of these is the effect of the component of the electric field (due to charge separation) parallel to the magnetic field. The importance of considering this field was pointed out by K. Watson.

We shall think in terms of the configuration illustrated in Fig. 6. The parallel part of the electric field will try to pull the charged particles out of the system along the **B** lines, but will be opposed by the parallel force on the magnetic moment, Eq. (8). When the electric field becomes large enough, it can pull the charges out of the system. This happens when the potential of the electric field is equal to the potential energy of the magnetic moment, $w_{\perp}$. As an estimate

$$eE_c\lambda \approx mv_l^2 \tag{50}$$

where E is the electric field of the separated charge, λ is the flute wavelength, and v_l is the Larmor velocity. When E_c reaches a value determined from Eq.

(50), additional charges will be drained off, and E_c will remain constant. The flute amplitude will then grow at a rate limited by

$$\dot{a} \leq c\frac{E_c}{B} \approx \frac{mv_l^2c}{eB\lambda\!\!\!^{-}} = \frac{r}{\lambda\!\!\!^{-}}v_l \tag{51}$$

where r is the Larmor radius. It is reasonable to suppose that the smallest permissible wavelength is about equal to the Larmor radius. Thus the flute growth rate is capable of attaining the full particle velocity, and the parallel fields do not really impede the small wavelength instabilities.

There is, however, another effect which limits the flute growth rate, namely, that when the amplitude exceeds the wavelength the linear theory no longer applies, and the growth rate is diminished. This is very similar to the situation in Taylor instability, where Taylor has shown, for example, that a bubble rises through a liquid eventually with constant velocity (even in the absence of viscous forces). The rate of rise, which is analogous to the flute growth rate, is

$$\dot{a} \approx \sqrt{g\lambda\!\!\!^{-}} \tag{52}$$

where g is the gravitational acceleration.

We shall show that the flute problem is identical to the hydrodynamic one, so that Taylor's result (52) can be carried over. Figure 8 shows the bubble in a rising frame of reference in which the flow is assumed stationary. Shaded regions represent plasma, and **B** is out of the paper and constant (low β). Only one wavelength is shown, but the configuration is assumed to repeat to the left and right.

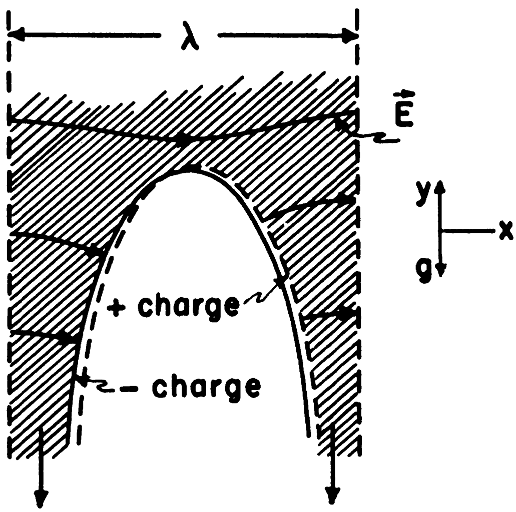

FIG. 8. The large-amplitude limit of the gravitational instability.

The gravitation drift

$$\mathbf{v}_g = -\frac{Mc}{e}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{53}$$

causes charges to appear on the surface of the plasma. This charge, together with the uniform upward motion of our coordinate system across the **B** field, produces an electric field, as illustrated in Fig. 8. The electric field then causes the plasma to flow with the velocity

$$\mathbf{v} = c\frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{54}$$

Since, in our coordinate system, conditions are assumed stationary, we may put

$$\mathbf{E} = -\nabla\phi. \tag{55}$$

In the interior of the plasma, then

$$\nabla^2\phi = 0.$$

Let χ be the harmonic function conjugate to the harmonic function ϕ; then according to (54)

$$\mathbf{v} = \frac{c}{B}\nabla\chi \tag{56}$$

and

$$\nabla \cdot \mathbf{v} = 0 = \nabla^2\chi \tag{57}$$

These are the equations of incompressible potential flow with velocity potential χ.

We still have to examine the boundary condition. The surface charge density σ is zero at the upper most point of the bubble, and changes along the surface, according to Eq. (53), at the rate (s is distance measured along the surface)

$$v\frac{\partial}{\partial s}\sigma = Nev_g\cos\theta = \frac{NMc}{B}g\cos\theta \tag{58}$$

where θ is the angle the surface makes with the y direction. Since **v** must be parallel to the surface, **E** must be perpendicular to the surface, and

$$\sigma = \frac{\epsilon E}{4\pi} \approx \frac{NMc^2}{B^2}E = \frac{NMc}{B}v. \tag{59}$$

Putting this result in Eq. (58), we find

$$\frac{\partial}{\partial s}\frac{v^2}{2} = g\cos\theta$$

or

$$\frac{\partial}{\partial s}\left(\frac{v^2}{2} + gy\right) = 0. \tag{60}$$

This is the Bernoulli equation expressing the constancy of pressure over the surface. The equivalence of the plasma problem and the hydrodynamic one is, therefore, complete, and we may use the result (52).

For our problem, the effective gravitational acceleration is

$$g = v_l^2/R$$

where R is the radius of curvature of the field lines. Thus we get a limit on $\dot{a}$

$$\dot{a} \leqq v_l \sqrt{\frac{\lambda\!\!\!^{-}}{R}}. \tag{61}$$

Presumably the most dangerous $\lambda\!\!\!^{-}$ will be that for which Eqs. (51) and (61) give the same $\dot{a}$, namely

$$\lambda\!\!\!^{-} = R^{1/3} r^{2/3}$$

for which

$$\dot{a} \leqq v_l (r/R)^{1/3} \tag{62}$$

Since r must be fairly small compared to R, we may expect that the maximum growth rate will be somewhat smaller than the Larmor (thermal) velocity.

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FINITE LARMOR RADIUS STABILIZATION OF "WEAKLY" UNSTABLE CONFINED PLASMAS*

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It is well known that the "modified" magnetohydrodynamic equations of motion follow from the exact plasma kinetic theory in the limit of small Larmor radius and low frequency, so that a magnetohydrodynamic prediction of instability is usually valid. However, in weakly unstable systems such as mirror machines, slowly rotating plasmas, large aspect ratio torii, etc., the expansion is no longer correct since the growth rate is very small, i.e., $(ka_i)^2$ may become comparable to ω/Ω_i so that ω must also be considered a small quantity. Here Ω_i is the ion cyclotron frequency, a_i the ion gyro-radius and k the wave number of the perturbation. We have studied several such cases—a plasma under gravity, a mirror machine, and a slowly rotating plasma. In all these cases the characteristic flute type instability is effectively stabilized if $(ka_i)^2 > \omega_H/\Omega$ where ω_H is the growth rate predicted by the hydrodynamic theory. It should be noted that present mirror experiments and fast-compression experiments are operating in the region $(a_i/r)^2 \approx \omega_H/\Omega_i$ and some detailed discussion of these cases will be made.

The dominant physical effect is the breakdown of the condition that ions and electrons move together across the magnetic field with characteristic velocity $\mathbf{v} = c\,\delta\mathbf{E}\times\mathbf{B}/B^2$, where $\delta\mathbf{E}$ is the perturbed electric field. Due to the finite ion Larmor radius the mean electric field seen by the ions is slightly different from the electrons so that their velocity across the field is different. This builds up a charge separation out of phase with the characteristic charge separation due to particle drifts which drive the flute type instability. If the above inequality is satisfied, the result is a stable oscillation.

1. Introduction

It is well known that the "modified" magnetohydrodynamic equations [1] of motion follow from the exact plasma kinetic theory in the limit of small Larmor radius and low frequency, so that a magnetohydrodynamic prediction of instability is usually valid. However, in weakly unstable systems such as mirror machines, slowly rotating plasmas, large aspect ratio torii, etc., the expansion is no longer correct since the growth rate ω is very small, i.e., $(ka_i)^2$ may become comparable to ω/Ω_i so that ω must also be considered a small quantity and terms $(ka_i)^2\,\Omega_i/\omega$ must be retained. Here Ω_i is the ion cyclotron frequency, a_i the ion gyro-radius, and k the wave number of the perturbation. We have studied several such cases—a plasma under gravity, a mirror machine, and a slowly rotating plasma. In all these cases the characteristic flute type instability [2] is effectively stabilized if $(ka_i)^2 > \omega_H/\Omega_i$ where ω_H is the growth rate predicted by the hydrodynamic theory. It should be noted that present mirror experiments [3] and fast-compression experiments [4] are operating in the region $(a_i/r)^2 \approx \omega_H/\Omega_i$ and some detailed discussions of these cases will be made.

The dominant physical effect is the breakdown of the condition that ions and electrons move together across the magnetic field with characteristic velocity $\mathbf{v} = c\,\delta\mathbf{E}\times\mathbf{B}/B^2$, where $\delta\mathbf{E}$ is the perturbed electric field. Due to the finite ion Larmor radius the mean electric field seen by the ions is slightly different from the electrons so that their velocity across the field is different. This builds up a charge separation out of phase with the characteristic charge separation due to particle drifts [2] which drives the flute type instability. If the above inequality is satisfied, the result is a stable oscillation.

This may be estimated as follows. Particles move across the field with velocity $\mathbf{v} = c\,\delta\mathbf{E}\times\mathbf{B}/B^2$. However, the electric field which should be used here is the mean electric field seen by the particle in its gyration. In particular, there is a difference between ion and electron velocity. Thus there is a separation current given roughly by $\mathbf{j}_F = a_i^2\,(\nabla^2\delta\mathbf{E}\times\mathbf{B})\,n_0 ec)B^2$. This is to be compared with the current which drives the instability in the hydrodynamic theory [2]. This current is equal to $\mathbf{j}_H = \mathbf{v}_D e\,\delta n$ where $\mathbf{v}_D$ is the equilibrium drift due to the destabilizing force, e.g., the gravitational drift $\mathbf{g}\times\mathbf{B}/\Omega B$; and δn is the perturbed number density due to the hydrodynamic motion $\delta n = -(c/i\omega)\,(\delta\mathbf{E}\times\mathbf{B}/B^2)\cdot\nabla n_0$. Recalling that $\omega_H = [g|\nabla n_0|/n_0]^{1/2}$ we see that $\mathbf{j}_F$ becomes comparable to $\mathbf{j}_H$ if $(ka_i)^2 \approx \omega_H/\Omega_i$.

In the following sections we study several situations of this kind. In all of these cases we treat the small amplitude stability problem by means of a self-

* Conference paper CN-10/170, presented by M. N. Rosenbluth. Discussion of this paper is given on page 208. Translations of the abstract are at the end of this volume of the Conference Proceedings.

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consistent solution of Maxwell's equation and the collisionless Boltzmann (Vlasov) equation, making use of an expansion which assumes $(ka_i) < 1$; $\omega/\Omega_i \approx (ka_i)^2$. These cases are further simplified by the assumption of low β (ratio of particle pressure to magnetic pressure); and by the fact that the equilibrium magnetic field is taken constant (to order β) with no shear or curvature. Other weakly unstable situations such as surface instabilities in pinches [5, 6] are not discussed in this paper although again important deviations from hydrodynamics will occur.

In Section 2 the gravitational instability problem is studied in detail. We discover that indeed the situation becomes almost stable for $(ka_i)^2 > \omega_H/\Omega_i$. There remains however a small residual instability of the Landau growth [7] type due to a resonant transfer with electrons drifting at the phase velocity of the wave.

In Section 3 the stability of a rotating cylindrical plasma is considered. Again the separation between ions and electrons is found to stabilize the motion. In this case the above condition leads to the stability requirement that the macroscopic rotation be not much greater than the rotation associated with the ion diamagnetic current. We also model the (unobserved!) flute instability caused by field line curvature by means of a radially outwards gravitational force. In this case we find the possibility of stabilizing all modes except one, a mode characterized by a constant electric field which it may be possible to stabilize by an external conductor.

2. Plasmas with plane geometry

The non-uniform plasma considered here is acted on by a magnetic field $\mathbf{B} = B_0 \mathbf{i}_z (1 + \varepsilon x)$, and a gravitational force $gm\mathbf{i}_x$. The equilibrium distribution function f_0 must satisfy the collisionless Boltzmann (Vlasov) equation and Maxwell's equations

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{e}{m}\left[\frac{\mathbf{v}\times\mathbf{B}}{c} + \frac{m}{e} g\,\mathbf{i}_x\right]\cdot\nabla_v f = 0$$

$$\frac{\partial f_0}{\partial t} = 0;\quad \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} = \frac{4\pi}{c}\sum_j e_j\int \mathbf{v} f_{0j}\mathrm{d}^3\mathbf{v} \tag{2.1}$$

$$\varrho = \sum_j e_j \int f_{0j}\mathrm{d}^3 v = 0\,.$$

Here j refers to particle species and is to be summed over values i and e for ions and electrons.

f_0 can be constructed from the constants of the motion: $\frac{1}{2}m_j v^2 - m_j g x$, $x + (v_y/\Omega_j)$. We choose a simple function

$$f_{0j} = \left(\frac{\alpha_j}{\pi}\right)^{3/2} n_{0j}\mathrm{e}^{-\alpha_j v^2}\left[1 - \varepsilon_j'\left(x + \frac{v_y}{\Omega_j}\right)\right]\mathrm{e}^{2\alpha_j g x}\,, \tag{2.2}$$

where we have defined $\Omega_j = e_j B/m_j c$. The parameter α_j is clearly $(v_{\text{thermal}})^{-2}$ and is related to a_j by $a_j = 1/\Omega_j\sqrt{\alpha_j}$. The parameter ε_j' is related to the magnetic field gradient through the second and third of Eqs. (2.1), which give

$$-\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}x} = \varepsilon_e' - 2\alpha_e g = \varepsilon_i' - 2\alpha_i g;\ \varepsilon\frac{B^2}{4\pi} = \sum_j n_j m_j\left[\frac{\varepsilon_j'}{2\alpha_j} - g\right]. \tag{2.3}$$

Thus ε'/ε is related to β, the ratio of particle pressure to magnetic field pressure, and in the cases we shall consider here the condition $\varepsilon' > \varepsilon$ obtains. Equation (2.2) thus describes an equilibrium slightly non-uniform plasma. We now examine the stability of this equilibrium in the presence of small perturbations. In the magnetohydrodynamic limit this equilibrium is unstable with growth rate $\sqrt{g\varepsilon'}$.

We write $f = f_0 + \delta f$, and use the Boltzmann Eq. (2.1) to obtain

$$\frac{\partial\delta f}{\partial t'} + \mathbf{v}\cdot\nabla\delta f + \frac{e}{m}\left[\frac{\mathbf{v}\times\mathbf{B}}{c} + \frac{m}{e} g\,\mathbf{i}_x\right]\cdot\nabla_v\delta f = -\frac{e}{m}\left[\delta\mathbf{E} + \frac{\mathbf{v}\times\delta\mathbf{B}}{c}\right]\cdot\nabla_v f_0\,, \tag{2.4}$$

where we have neglected non-linear terms such as $\delta\mathbf{E}\cdot\nabla_v\delta f$. We can formally solve this equation by introducing the parameter t, and defining the transformation $t', \mathbf{x}, \mathbf{v} \to t, \mathbf{x}'(t), \mathbf{v}'(t)$ by

$$t' = t,\quad \frac{\mathrm{d}\mathbf{x}'}{\mathrm{d}t} = \mathbf{v}'(t),\quad \frac{\mathrm{d}\mathbf{v}'}{\mathrm{d}t} = \Omega(\mathbf{v}\times\mathbf{i}_z)(1 + \varepsilon x') + g\,\mathbf{i}_x\,. \tag{2.5}$$

We observe that these equations describe orbits of charged particles moving in the unperturbed force fields $\mathbf{F} = (e/c)\,\mathbf{v}\times\mathbf{B} + mg\mathbf{i}_x$. In terms of the transformation (2.5), the Boltzmann equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t'}\delta f(x', v', t') = -\frac{e}{m}\left[\delta\mathbf{E} + \frac{\mathbf{v}'\times\delta\mathbf{B}}{c}\right]\cdot\nabla_{v'} f_0 \tag{2.6}$$

and choosing the boundary conditions $\mathbf{x}'(t=0) = \mathbf{x}$, $\mathbf{v}'(t=0) = \mathbf{v}$ we have

$$\delta f(x, v, t=0) = -\frac{e}{m}\int_{-\infty}^{0}\mathrm{d}t\left[\delta\mathbf{E} + \frac{\mathbf{v}'\times\delta\mathbf{B}}{c}\right]\cdot\nabla_{v'} f_0\,. \tag{2.7}$$

We now assume that all perturbed quantities have time-space dependence $\mathrm{e}^{\mathrm{i}\omega t} g(\mathbf{x})$. This procedure will yield the same results as the rigorous procedure using Laplace transforms if we assume that ω has a small negative imaginary part, such that at $t = -\infty$, δf vanishes. Combining Eq. (2.7) with Maxwell's equation then gives a dispersion relation for ω. We assume that the perturbation is associated only with a longitudinal electrostatic field $\delta\mathbf{E} = -\nabla\Psi$. In fact, of course, the longitudinal waves are coupled to the transverse waves. The coupling can be shown to be small for small values of β = particle pressure/magnetic field pressure, and our approximation is valid only in that limit. This decoupling must occur for unstable waves, since a transverse electric field would lead to a modification of the magnetic field and an increase of field energy which in the low β limit would exceed the change in particle energy. In a separate paper we explicitly examine transverse modes and demonstrate that the present results are indeed valid for small β. Our dispersion relation is then

$$\nabla^2 \Psi = -\frac{4\pi}{c}\sum e_j \int \delta f_j \, d^3 v$$
$$= -\frac{4\pi}{c}\sum \frac{e_j^2}{m_j}\int d^3 v \int_{-\infty}^{0} dt \{\nabla \Psi(x',t) \cdot \nabla_{v'} f_0(x',t)\}. \quad (2.8)$$

The lowest order solutions to the orbit equations (2.5) are

$$\begin{aligned} v_y &= v_\perp \sin(\theta - \Omega t) + v_D \\ y &= \frac{v_\perp}{\Omega}\cos(\theta - \Omega t) - \frac{v_\perp}{\Omega}\cos\theta + v_D t \\ v_D &= \frac{1}{2}\frac{\varepsilon v_\perp^2}{\Omega} - \frac{g}{\Omega} \\ \frac{d}{dt}\left(x + \frac{v_y}{\Omega}\right) &= \frac{d}{dt}(v^2 - 2gx) = 0, \end{aligned} \quad (2.9)$$

with similar expressions for x, v_x, and v_z. We have explicitly indicated two constants of the motion allowed by the orbit equation, $x + (v_y/\Omega)$ and $(v^2 - 2gx)$. Thus $f_0(x', v') = f_0(x, v)$, and $e^{-\alpha(v')^2} e^{2\alpha g x'} = e^{-\alpha v^2} e^{2\alpha g x}$.

An interesting and important feature of these orbits is the drift velocity v_D, which has two parts, the $\varepsilon v^2/2\Omega$ field gradient drift, and the $-g/\Omega$ gravitational drift. Since $\nabla|B|$ and g are both vectors pointing in the x-direction and **B** is in the z-direction, both of those drifts are in the y-direction. These particle drifts represent a possible source of instability [2], causing a pile-up of space charge along the instability flutes. With this in mind we solve Eq. (2.8) for $\Psi \approx e^{i\omega t + iky}$. For $k > \varepsilon'$ the x dependence of the perturbation is weak and may be neglected [8]. In the next Section we retain the complete space dependence.

With the simplifications, using the explicit orbits of Eq. (2.9), the integrals in Eq. (2.8) can be evaluated. Thus

$$\nabla_v f_{0j} = -2\alpha_j \mathbf{v} f_{0j} - \frac{\varepsilon_j'}{\Omega_j}\left(\frac{\alpha_j}{\pi}\right)^{3/2} n_{0j} e^{-\alpha_j [v^2 - 2gx]} \mathbf{i}_y .$$

We note that

$$\mathbf{v} \cdot \nabla \Psi = \frac{d\Psi}{dt} - i\omega\Psi; \; \mathbf{i}_y \cdot \nabla \Psi = i k \Psi,$$

where d/dt means total derivative along the particle orbit. Noting that f_0 and $v^2 - 2gx$ are constants of the motion, we may write the time integral in Eq. (2.8) in the form

$$\int_{-\infty}^{0} dt \nabla \Psi \cdot \nabla_v f_0 = -2\alpha_j f_0 \Psi + i\left\{2\alpha_j\omega - \frac{\varepsilon_j' k}{\Omega_j}\right\} f_0 \int_{-\infty}^{0} dt \, e^{ik(y'-y) + i\omega t}.$$

This expression neglects a small term of order ε'^2 which may easily be shown to vanish in a subsequent integration.

Using Eq. (2.9) we have

$$e^{ik(y'-y)} = \exp\left\{ik\left[\frac{1}{2}\frac{\varepsilon v_\perp^2}{\Omega} - \frac{g}{\Omega}\right]t\right\}$$
$$\sum_{l,m} J_l\left(\frac{kv_\perp}{\Omega}\right) J_m\left(\frac{kv_\perp}{\Omega}\right) e^{i(l-m)\pi/2} e^{i(l-m)\theta} e^{-il\Omega t}.$$

The integration over the azimuthal velocity space angle, θ, t, and v_z is now easily done to give the dispersion relation:

$$-k^2 = 8\pi e^2 n\left(\frac{\alpha_i}{M} + \frac{\alpha_e}{m}\right)$$
$$-4\pi\sum_{j,l}\int \frac{e^{-\alpha_j v_\perp^2} 2\alpha_j v_\perp dv_\perp [2\alpha_j\omega - (k\varepsilon_j/\Omega_j)] J_l^2(kv_\perp/\Omega_j)}{(\omega + l\Omega_j + kv_D)} . \quad (2.10)$$

In obtaining Eq. (2.10) we have set $x = 0$ in expressions like $e^{2\alpha g x}$, consistent with setting $\Psi = e^{i\omega t + iky}$. It is obvious that all the terms in the sum over Bessel functions may be ignored except the $l = 0$ terms, higher terms being smaller by a factor $(ka_i)^2(\omega/\Omega_i)^2 \ll 1$. We reiterate that our calculation assumes ka_i, $\varepsilon' a_i < 1$, as in the magnetohydrodynamic limit, and that the importance of higher orders in ka_i arises not from large values of ka_i but from extremely small values of the parameter ω/Ω for "weakly unstable" situations. In particular, in Eq. (2.10) we treat $k\varepsilon'/\Omega_j$ as of the same order as $2\alpha_j\omega$.

Expanding the Bessel function and neglecting the Debye length compared to Larmor radius, we have:

$$0 = F(\omega) = 2\sum_j \frac{\alpha_j}{m_j}$$
$$-\sum_j\left(2\alpha_j\omega - \frac{k\varepsilon_j'}{\Omega_j}\right)\frac{1}{m_j}\int_0^\infty \frac{e^{-x}[1 - (k^2 x/2\alpha_j\Omega_j^2)]\,dx}{\omega - (kg/\Omega_j) + (k\varepsilon x/2\alpha_j\Omega_j)} . \quad (2.11)$$

The singularities of the integral are to be interpreted in the usual way, passing to the limit as ω has a small negative imaginary part [7]. To study the stability of the system we must determine whether there exist solutions ω of Eq. (2.11) with negative imaginary part. This question can be systematically answered by studying the behavior of $F(\omega)$ in the complex plane as ω goes from $-\infty$ to $+\infty$ along the real axis, the number of unstable roots being given by the number of times the origin is encircled by the curve $F(\omega)$. We omit the details and give only the result—if the density decreases in the direction of gravity, i.e., $\varepsilon' > 2\alpha g > 0$, there is one unstable root. We will subsequently find this root by approximate means.

As we are primarily concerned with the case of low β, i.e., $\varepsilon'/\varepsilon \gg 1$, we disregard the last term in the denominator of Eq. (2.11) except for the imaginary part which arises from the singularity of the integral. This small term will eventually yield a slow Landau damping or growth of our solutions.

The integrals may now be performed to give

$$0 = \sum_j \left\{ \frac{\alpha_j}{m_j} - \left(\frac{\alpha_j \omega}{m_j} - \frac{k \varepsilon_j'}{2 \Omega_j m_j} \right) \left[\frac{1 - (k^2/2\alpha_j \Omega_j^2)}{\omega - (kg/\Omega_j)} + i\gamma_j \right] \right\}, \quad (2.12)$$

where

$$\gamma_j = \frac{\pi}{k\varepsilon/2\alpha_j |\Omega_j|} \exp\left[-\left(\omega - \frac{kg/\Omega_j}{k\varepsilon/2\alpha_j \Omega_j} \right) \right]$$

or $\gamma_j = 0$ depending on whether the exponent is negative or positive. Equation (2.12) may now easily be reduced to a quadratic and solved. First we neglect the small terms γ_j, and terms of order m_e/m_i. Using Eq. (2.3) we find

$$\omega = \frac{1}{2} \left\{ \left(\frac{k\varepsilon_i'}{2\alpha_i \Omega_i} \right) \pm \left[\left(\frac{k\varepsilon_i'}{2\alpha_i \Omega_i} \right)^2 - 4 g \varepsilon_e' \right]^{\frac{1}{2}} \right\}. \quad (2.13)$$

This is to be compared with the hydrodynamic result $\omega_H = \pm i \sqrt{g\varepsilon'}$. We see that now the system is "stable" if

$$\frac{k\varepsilon'}{2\alpha_i \Omega_i} > 2\omega_H$$

or

$$(k a_i)(\varepsilon' a_i) > \frac{4\omega_H}{\Omega_i}. \quad (2.14)$$

Thus we see that for weakly unstable systems the hydrodynamic approximation may break down even for $ka_i \ll 1$. It is interesting to note that the larger the density gradient, the more stable the systems. This may possibly be relevant in relation to the van Allen radiation belts. We also note that the condition (Eq. (2.14)) is considerably more favorable to stability than the requirement that the growth period be short compared to the time required to drift a wavelength.

Finally we return to Eq. (2.12) to examine the effect of the small imaginary terms in the dispersion relation. It is easily seen that only the electron terms contribute in the neighborhood of the roots (Eq. (2.13)) and that the negative sign in Eq. (2.13) yields the unstable root. In the "stable" limit $k\varepsilon'/2\alpha_i\Omega_i \gg 2\omega_H$ we have for the exponential dependence of the growth rate

$$\begin{aligned} \mathrm{Im}(\omega) &\propto e^{-4(g/\varepsilon k^2)\alpha_i \Omega_i \alpha_e |\Omega_e|} \\ &\propto e^{-(8/\beta)(T_i/T_e) g \alpha_i/\varepsilon' k^2 a_i^2}. \end{aligned} \quad (2.15)$$

Thus at wavelengths comparable to the ion Larmor radius and for moderate β even this overstability may lead to sizable growth although it is not clear how seriously to take this in view of our assumptions β, $ka_i \ll 1$. For example, we might require for all wavelengths for which our calculation is valid, i.e., $1/a_i > k > \varepsilon'$, that the system be stable in the sense of satisfying Eq. (2.14) and having the exponent in Eq. (2.15) greater than 8. This effective stability condition is

$$\left(\frac{\varepsilon' a_i}{4} \right)^2 > \frac{\alpha g}{\varepsilon'} > \beta \frac{T_e}{T_i}.$$

3. Confined plasma with cylindrical symmetry

In this Section we study the stability of a cylindrical plasma confined by a longitudinal magnetic field. Taking advantage of the results of the preceding Section, we are able to neglect the small drifts due to magnetic field gradients and expand the denominators which occur in the various integrals over the particle distribution function. On the other hand, in this Section we will do the calculation self-consistently, taking properly into account the radial dependence of the disturbances rather than assuming a localized perturbation.

The first problem we treat is that of a uniformly rotating plasma. It is easy to show in the magnetohydrodynamic approximation that such a plasma is unstable to long-wave length perturbations with a growth rate equal to $\sqrt{m-1}\, W$, where m is the azimuthal wave number and W the angular velocity. From the microscopic viewpoint the rotation of the plasma may be split into two parts, that associated with the diamagnetic current and that produced by a radial electric field. The diamagnetic current is carried by an azimuthal velocity of the particles equal to their mean velocity multiplied by the ratio of the Larmor radius to plasma radius. Hence $W_{\mathrm{diamagnetic}} = (a_i/r)^2 \Omega_i$. We see that the growth rate associated with the diamagnetic rotation is just such as to fall within our definition of weak instability. Thus the hydromagnetic approximation is inadequate for discussing the possible rotational instability associated with plasma diamagnetism. We proceed to calculate the stability of such a diamagnetic plasma allowing also for an electric field induced rotation of the same order as the diamagnetic rotation. In what follows we treat ω/Ω_i, $W/\Omega_i \approx O(a_i/r)^2$.

The plasma is assumed to be infinite in the z-direction. All perturbations are proportional to $\exp i(\omega t + kz + m\varphi)$; the present calculations will however be restricted to perturbations for which $k = 0$. A further restriction to be imposed is that the plasma density is so low that the particles are unable to perturb the magnetic field. In addition to the usual expansion parameters of the modified magnetohydrodynamic approximation [1], $\beta = 8\pi P/B^2$ where P is the plasma pressure will be considered to be a small quantity and calculations will be carried out to the lowest order in β; i.e., in the limit $\beta \to 0$. In this limit we can take the unperturbed magnetic field B to be constant and in the z-direction. Moreover, since the perturbed magnetic field $\delta \mathbf{B} = 0$ in this limit, the perturbed electric field $\delta \mathbf{E} = \nabla \delta \Psi$ is the gradient of a potential.

3.1. Stationary state of the plasma

The initial distribution function $f_j(\mathbf{x}, \mathbf{v})$ satisfies the Vlasov equation

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla f_j + \frac{e_j}{m_j} \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = 0, \quad (3.1)$$

where $\mathbf{B} = [0, 0, B]$ and B is constant. $\mathbf{E} = -(WB/c)[x, y, 0]$ is a radial electric field; W is the constant angular velocity of guiding centers which is assumed to be a small quantity such that $W/\Omega_j = O(\bar{a}_j/r_0)^2$ where $\Omega_j = e_j B/m_j c$ is the Larmor frequency and $\bar{a}_j$

is the mean Larmor radius for particles of species j; r_0 is the radius of the plasma.

The unperturbed orbit equations are

$$\ddot{x} = -\Omega_j [Wx - \dot{y}]$$
$$\ddot{y} = -\Omega_j [Wy + \dot{x}] \tag{3.2}$$
$$\ddot{z} = 0,$$

where $\dot{x} = dx/dt = v_x$ etc. The constants of the motion are the energy

$$\varepsilon_j = \frac{m_j}{2}(v^2 + \Omega_j W r^2), \tag{3.3}$$

and the canonical angular momentum

$$L_j = m_j (x v_y - y v_x) + \frac{e_j}{c} r A_\varphi . \tag{3.4}$$

$A_\varphi = Br/2$ is the vector potential of the magnetic field and $r^2 = x^2 + y^2$.

The solution of Eq. (3.1) is an arbitrary function of the constants of the motion ε_j and L_j. A specific function will be assumed of the form

$$f_j(\mathbf{x}, \mathbf{v}) = \frac{A_j}{(2\pi v_j^2)^{3/2}} \exp\left(-\frac{\varepsilon_j + \omega_j L_j}{m_j v_j^2}\right). \tag{3.5}$$

The particle density $n_j(r) = \int f_j d\mathbf{v}$ must be independent of species in the sense that $\delta n/n \ll (\bar{a}_j/r_0)^2$, since we are looking for effects of this order.

As it is also true that $\delta n/n \approx (L_D/r_0)^2$ where L_D is the Debye length, we must require the Debye length to be much less than the ion Larmor radius or

$$1 \gg \beta > \frac{v_i^2}{c^2}. \tag{3.6}$$

After integrating Eq. (3.5) over velocity space the following result is obtained for particle density

$$n(r) = \frac{N}{\pi r_0^2} \exp -\left(\frac{r}{r_0}\right)^2, \tag{3.7}$$

where $N = \pi r_0^2 A_j$ is the number of particles per unit length and A_j is independent of species. The radius of the plasma is r_0 where

$$\frac{1}{r_0^2} = \frac{1}{2 v_j^2}\{\Omega_j (W + \omega_j) - \omega_j^2\}.$$

This determines the parameter ω_j as

$$\omega_j = \frac{\Omega_j}{2}\left\{1 - \sqrt{1 - \frac{4}{\Omega_j}(\overline{W}_1 - W)}\right\}$$
$$= \overline{W}_j - W + \frac{1}{\Omega_j}(\overline{W}_j - W)^2 + O\left(\frac{a_j}{r_0}\right)^3. \tag{3.8}$$

In this equation $\bar{a}_j = v_j/\Omega_j$ and $\overline{W}_j = 2(\bar{a}_j/r_0)^2 \Omega_j$. Eq. (3.5) can be expressed as

$$f_j(\mathbf{x}, v) = \frac{n(r)}{(2\pi v_j^2)^{3/2}} \exp\left\{-\left[\frac{v^2}{2 v_j^2} + \frac{\omega_j}{v_j^2}(x v_y - y v_x) + \frac{\omega_j^2 r^2}{2 v_j^2}\right]\right\}. \tag{3.9}$$

This provides a complete description of the stationary state to be considered.

We note that $\overline{W}_j$ is the angular frequency associated with the motion of the particles carrying the plasma diamagnetic current. It provides a natural unit of angular frequency and we are interested in studying cases where $W/\overline{W} \approx 1$.

3.2. Solution of the unperturbed orbit equations

With the substitution $\xi = x + iy$, Eqs. (3.2) simplify to

$$\ddot{\xi} + i\Omega_j \dot{\xi} + \Omega_j W \xi = 0$$
$$\ddot{z} = 0. \tag{3.10}$$

The solution is

$$\xi(t) = a e^{i\omega_a t} + b e^{i\omega_b t}$$
$$\dot{z} = v_z, \tag{3.11}$$

where

$$\omega_b = -\frac{\Omega_j + \sqrt{\Omega_j^2 + 4\Omega_j W}}{2} = W\left\{1 - \frac{W}{\Omega_j}\dots\right\}$$
$$\omega_a = -\Omega_j\left\{1 + \frac{W}{\Omega_j} - \left(\frac{W}{\Omega_j}\right)^2 \dots\right\}.$$

The complex constants a and b can be identified by the initial conditions

$$\xi(0) = r e^{i\varphi} = a + b$$
$$\dot{\xi}(0) = v_\perp e^{i\theta} = i[\omega_a a + \omega_b b]. \tag{3.12}$$

The solution may be expressed as

$$\xi e^{-i\omega_b t} = b + a e^{i(\omega_a - \omega_b)t}.$$

Since $b \cong r$, $a = v_\perp/\Omega_j$, $\omega_b \cong W$ and $\omega_a - \omega_b \cong -\Omega_i$ the unperturbed orbit consists approximately of a conventional Larmor orbit about a guiding center a distance $|b|$ from the origin which rotates with angular velocity W.

3.3. Linearized stability analysis

The linearized Vlasov equation is

$$\frac{\partial}{\partial t}\delta f_j + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\delta f_j + \frac{e_j}{m_j}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\delta f_j$$
$$= -\frac{e_j}{m_j}\left(\delta\mathbf{E} + \frac{1}{c}\mathbf{v}\times\delta\mathbf{B}\right)\cdot\frac{\partial f_j}{\partial \mathbf{v}}. \tag{3.13}$$

We consider only the limit $\beta \to 0$ in which case $\delta\mathbf{B} \to 0$ and $\delta\mathbf{E} = \nabla\delta\Psi$. Perturbations will be considered of the form $\delta\Psi = \Psi(r) \exp i(\omega t + m\varphi)$.

The first step is to integrate Eq. (3.13) along the unperturbed orbits:

$$\delta f_j = -\frac{e_j}{m_j}\int_{-\infty}^{t} dt' \left(\delta\mathbf{E}\cdot\frac{\partial f_j}{\partial \mathbf{v}}\right)_{t'}.$$

Making use of Eq. (3.9) for f_j

$$\left(\delta\mathbf{E}\cdot\frac{\partial f_j}{\partial \mathbf{v}}\right) = -\frac{f_j}{v_j^2}\{\mathbf{v}\cdot\nabla\delta\Psi + \omega_j(x\delta E_y - y\delta E_x)\}.$$

f_j is a function of the constants of the motion of the unperturbed orbits and as such can be taken outside the integral. (d/dt) $\delta\Psi = \mathrm{i}\omega\delta\Psi + \mathbf{v}\cdot\nabla\delta\Psi$ and $x\delta E_y - y\delta E_x = \mathrm{i}m\,\delta\Psi$. It is assumed that ω has a small negative imaginary part* so that $\delta\Psi(-\infty) = 0$. With these substitutions, and making use of Eq. (3.11) for $\xi(t)$, the following result obtains for δf_j,

$$\delta f_j = \frac{e_j f_j}{m_j v_j^2} \exp \mathrm{i}(\omega t + m\varphi) \left\{ \Psi(r) - \mathrm{i}(\omega - m\omega_j) \int_{-\infty}^{0} \frac{\Psi(|\xi|)}{|\xi|^m} \xi^m \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \right\}. \quad (3.14)$$

In order to carry out the indicated time integration it is necessary to make an expansion in powers of $\lambda = \bar{a}_j/r_0$. It is assumed that ω/Ω_j, $W/\Omega_j = \mathrm{O}(\lambda^2)$ and we consider $v_\perp/r\Omega_j$ to be of order λ. A Taylor series is employed for $\Psi(|\xi|)$, i.e.,

$$|\xi(t)| = r + \delta r(t) \text{ and } \Psi(|\xi(t)|)$$
$$= \Psi(r) + \delta r(t)\,\Psi'(r) + \frac{[\delta r(t)]^2}{2!}\Psi''(r)\ldots$$
$$= (\Psi - r\Psi' + \frac{r^2}{2}\Psi'') + \frac{|\xi(t)|}{r}(r\Psi' - r^2\Psi'') + \frac{|\xi(t)|^2}{2}\Psi''.$$

After some algebra we find the general term of the expansion is of the form

$$\xi^m \frac{|\xi|^{n-m}}{r^n} = \exp[\mathrm{i}m(\varphi + \omega_b t)] \left\{ 1 - n(W/\Omega_j) + \frac{(n^2 - m^2)}{2}\left(\frac{v_\perp}{r\Omega_j}\right)^2 - \frac{\mathrm{i}m}{r^2\Omega_j}(xv_x + yv_y) + \frac{n}{r^2\Omega_j}(xv_y - yv_x) \right\}. \quad (3.15)$$

In deriving (3.15) we have of course only retained terms of the proper order; $(v/r\Omega_j)^2$, W/Ω_j. We have moreover dropped terms containing factors $\exp \mathrm{i}(\omega_a - \omega_b)t$ and $v_x^2 - v_y^2$ since they will become of higher order after subsequent integration.

After carrying out the time integration in Eq. (3.14) the result is

$$\delta f_j = \frac{e_j}{m_j v_j^2} f_j \exp[\mathrm{i}(\omega t + m\varphi)] \Bigg(\Psi(r) - \frac{(\omega - m\omega_j)}{(\omega + m\omega_b)} \times \Bigg\{ \Psi(r) \left[1 - \frac{\mathrm{i}m}{r^2\Omega_j}(xv_x + yv_y) - \frac{m^2}{2}\left(\frac{v_\perp}{r\Omega_j}\right)^2 \right] + r\Psi'(r) \left[\frac{(xv_y - yv_x)}{r^2\Omega_j} - \frac{W}{\Omega_j} + \frac{1}{2}\left(\frac{v_\perp}{r\Omega_j}\right)^2 \right] + \frac{r^2}{2}\Psi''(r)\left(\frac{v_\perp}{r\Omega_j}\right)^2 \Bigg\}. \quad (3.16)$$

It is now possible to calculate the perturbation of macroscopic variables. For example, the perturbation of charge density is

$$\delta\varrho_e = \sum_j e_j \int \delta f_j \mathrm{d}\mathbf{v} = \exp[\mathrm{i}(\omega t + m\varphi)] \sum_j \frac{n(r) e_j^2}{m_j v_j^2} \Bigg(\Psi(r) - \frac{\omega - m\omega_j}{\omega + m\omega_b} \Bigg\{ \Psi(r)\left[1 - m^2\left(\frac{\bar{a}_j}{r}\right)^2\right] - r\Psi'(r)\left[\frac{\omega_j + W}{\Omega_j} - \left(\frac{\bar{a}_j}{r}\right)^2\right] + r^2\Psi''(r)\left(\frac{\bar{a}_j}{r}\right)^2 \Bigg\} \Bigg). \quad (3.17)$$

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* $\delta\psi = \lim_{\gamma\to 0} \psi(r) \exp \mathrm{i}\, m\,\varphi \exp(\mathrm{i}\omega + \gamma)\,t$ means that the perturbation is switched on adiabatically at $t = -\infty$. The same procedure obtains if we let $p = \mathrm{i}\,\omega + \gamma$ be the Laplace transform variable.

To complete the expansion procedure substitute ω_j from Eq. (3.8) and making use of Eq. (3.11)

$$\frac{1}{\omega + m\omega_b} = \frac{1}{\omega + mW}\left\{1 + \frac{mW^2/\Omega_j}{\omega + mW}\ldots\right\}$$

After carrying out the indicated multiplication

$$4\pi\delta\varrho_e = \exp[\mathrm{i}(\omega t + m\varphi)] \sum_j \frac{1}{L_j^2}\left\{ \Psi\left[\frac{\overline{W}_j}{\bar{\omega}W} + \frac{\overline{W}_j(\overline{W}_j - 2W)}{W\Omega_j\bar{\omega}} - \frac{\overline{W}_j}{\Omega_j\bar{\omega}^2}\right] + \left[1 - \frac{\overline{W}_j}{\bar{\omega}W}\right]\left[\bar{a}_j^2\left(\frac{m^2}{r^2}\Psi - \frac{\Psi'}{r} - \Psi''\right) + r\Psi'\frac{\overline{W}_j}{\Omega_j}\right]\right\},$$

where $\bar{\omega} = (\omega + mW)/mW$, $\overline{W}_j = 2(\bar{a}_j/r_0)^2\Omega_j$ and $1/L_j^2 = 4\pi n(r)e_j^2/m_j v_j^2$. The first term gives zero after summing because $\overline{W}_j/L_j^2 = 8\pi c n(r) e_j/r_0^2 B$ which is equal and opposite for electrons and ions. In the remaining terms the contributions from the ions dominate so that we can omit the sum and finally obtain

$$4\pi\delta\varrho_e = \left(\frac{\bar{a}_i}{L_i}\right)^2\left[\frac{\overline{W}_i}{\bar{\omega}W} - 1\right]\exp[\mathrm{i}(\omega t + m\varphi)] \left[\Psi'' + \frac{\Psi'}{r} - \frac{m^2}{r^2}\Psi - \frac{2}{r_0^2}(r\Psi' - \nu\Psi)\right] \quad (3.18)$$

where

$$\nu = -\frac{(1/\bar{\omega}) + (\overline{W}_i/W) - 2}{\bar{\omega} - (\overline{W}_i/W)}.$$

$\delta\psi$ satisfies the differential equation $\nabla^2\delta\psi = +4\pi\delta\varrho_e$. It has already been assumed that $(L_i/\bar{a}_i)^2 \ll \lambda^2 \ll 1$ so that ψ must satisfy

$$\Psi'' + \frac{\Psi'}{r} - \frac{m^2}{r^2}\Psi - \frac{2}{r_0^2}(r\Psi' - \nu\Psi) = \mathrm{O}\left(\frac{L_i}{\bar{a}_i}\right)^2 \cong 0. \quad (3.19)$$

If the following substitutions are made

$$x = (r/r_0)^2, \quad \mu = \frac{1+\nu}{2}, \quad s = \frac{m}{2}, \quad \Psi = \frac{y(x)}{\sqrt{x}}\exp\frac{x}{2},$$

this equation becomes

$$y'' + \left\{-\frac{1}{4} + \frac{\mu}{x} + \frac{1/4 - s^2}{x^2}\right\} y = 0, \quad (3.20)$$

which is Whittaker's equation [9]. The solutions are Whittaker functions $W_{\mu,s}(x)$ and $W_{-\mu,s}(-x)$. The asymptotic form for large $|x|$ is

$$\lim_{|x|\to\infty} W_{\mu s}(x) = x^\mu \mathrm{e}^{-\frac{x}{2}}\left\{1 + \sum_{l=1}^{\infty}\frac{1}{l!\,x^l}\prod_{l'=1}^{l}\left[s^2 - \left(\mu + \frac{1}{2} - l'\right)^2\right]\right\}. \quad (3.21)$$

If $\mu - \frac{1}{2} - s = n$ where n is an integer the sum terminates at $l = n + 2s$. The asymptotic form for ψ from this solution is $\lim \psi(r) = r^{m+2n}$ as $r \to \infty$. The second solution from $W_{-\mu,s}(-x)$ diverges like $\exp(r/r_0)^2$ so that boundary conditions at large r can be satisfied with the first solution dominating. This is an eigenvalue problem with the eigenvalues for ν or μ

$\nu = 2\mu - 1 = m + 2n$, and therefore

$$\nu\bar{\omega} = \left\{\frac{\overline{W}_i}{2W}(\nu - 1) + 1 \pm \sqrt{\left[\frac{\overline{W}_i}{2W}(\nu - 1) + 1\right]^2 - \nu}\right\}, \quad (3.22)$$

where we have made use of Eq. (3.18). If $n = 0$, $\omega = 2W - \overline{W}_i$ for $m = 0$, and $\omega = 0$ for $m = 1$. For $\nu > 1$ the condition for stability is

$$\left(\frac{\overline{W}_i}{2W}(\nu - 1) + 1\right)^2 > \nu, \text{ or } \frac{1 + \sqrt{\nu}}{2} > \frac{W}{\overline{W}_i} > \frac{1 - \sqrt{\nu}}{2}. \quad (3.23)$$

A symmetric form of this criterion is obtained by noting that the macroscopic velocity V_φ in the initial state is given by

$$\varrho_m V_\varphi = \sum_j \frac{m_j}{r}\int f_j (xv_y - yv_x)\,\mathrm{d}\mathbf{v}$$

so that

$$V_\varphi = r(W - \overline{W}_i), \quad (3.24)$$

and the stability criterion for $m > 1$ is

$$\frac{\sqrt{\nu}}{2} > \frac{1}{2\overline{W}_i}\left(\frac{V_\varphi}{r} + W\right) > -\frac{\sqrt{\nu}}{2}. \quad (3.25)$$

Thus we see that the rotation associated with the plasma diamagnetic currents is not sufficient to induce instability by itself. However, if the rotation produced by a radial electric field exceeds this natural rotation by an appreciable factor, then the hydrodynamic instability occurs.

3.4. Application to the fast B_z-compression (Scylla)

It has been observed [4] that the plasma formed in a fast B_z-compression develops an $m = 2$ instability after a quiescent period of about 5 μs. If the radial displacement is proportional to $\exp i\,[\omega t + m\varphi + kz]$ and only the $m = 2$ mode grows, the plasma should appear to fission into two parts which have an apparent rotational frequency of $\mathrm{d}\varphi/\mathrm{d}t = -\mathrm{Re}\,\omega/2$. This is in fact observed and direct measurements from streak camera photographs give $\mathrm{Re}\,\omega/2 = 10^7\,\mathrm{s}^{-1}$. The observed rotation is in the same sense as ion orbits in the external field.

On the basis of Kolb's measurements of ion temperature ($\Theta_i = 2$ keV), plasma radius ($r_0 = 0.8$ cm) and magnetic field (B = 50 kG) we can estimate $\overline{W}_i = 2(a_i/r_0)^2\Omega_i \cong \frac{1}{4} \times 10^8\,\mathrm{s}^{-1}$. For an $m = 2$ instability, Eq. (3.22) gives

$$\mathrm{Re}\,\omega = -\left(W - \frac{\overline{W}_i}{2}\right) = 2 \times 10^7\,\mathrm{s}^{-1},$$

from which we deduce that $W/\overline{W}_i = -0.3$. The stability predictions of the previous analysis are that $m = 0$, 1 should be stable and from Eq. (3.23) the stability limits are

$$1.207 > \frac{W}{\overline{W}_i} > -0.207 \text{ for } m = 2,$$

$$1.366 > \frac{W}{\overline{W}_i} > -0.366 \text{ for } m = 3.$$

The experimental value of $W/\overline{W}_j = -0.3$ is in agreement with the observed fact that only the $m = 2$ mode is unstable, although the quantitative agreement with theory is not to be taken seriously in view of the idealized theory and experimental inaccuracies. In another paper [4] it has been shown that the redistribution of currents associated with the decay of trapped magnetic field should produce rotations $W/\overline{W}_i \approx 1$ and hence lead to $m = 2$ instabilities.

3.5. Application to mirror machines

The geometry of a mirror machine makes an exact treatment of stability somewhat unpleasant. The present calculations for an infinite cylinder may be adapted to the mirror machine if we include the essential feature of the curvature of the field lines. It is well known [2] that the effect is similar to that of a gravitational field and produces flute instabilities with a growth rate of the order of $\sqrt{mg/r_0}$ where the effective gravitational constant is $g \sim v_i^2/R$, where R is the average radius of curvature of the field lines and v_i is the thermal speed of the ions. For sufficiently large

$$R \sim r_0^3/\bar{a}_i^2, \quad (3.26)$$

the instability is weak and $\sqrt{gk_0}/\Omega_i \sim (a_i/r_0)^2$ so that the magnetohydrodynamic approximation breaks down. This case is of some practical significance since in most experiments $\bar{a}_i/r_0$ is not very large and $r_0/R \ll 1$ and can be treated by a slight extension of the present calculations.

In order to take account of the fact that the curvature of the field lines increases with r we assume an equivalent gravitational force of the form

$$\left[\frac{m_i v_i^2}{R}\frac{x}{r_0},\ \frac{m_i v_i^2}{R}\frac{y}{r_0},\ 0\right].$$

The unperturbed equations of motion are then the same as Eq. (3.2) if we simply replace W by $W_i = W - v_i^2/\Omega_i r_0 R$. The entire previous analysis can be carried through except that the final expansion produces an additional term. Eq. (3.14) is obtained with the new definition of ν:

$$\nu = \frac{\left\{\frac{1}{\bar{\omega}}\left(1 + \frac{v_i^2}{W^2 r_0 R}\right) + \frac{\overline{W}_i}{W} - 2\right\}}{\bar{\omega} - \frac{\overline{W}_i}{W}}, \quad (3.27)$$

or

$$\nu\bar{\omega} = \frac{\overline{W}_i}{2W}(\nu - 1) + 1 \pm \sqrt{\left[\frac{\overline{W}_i}{2W}(\nu - 1) + 1\right]^2 - \nu\left[1 + \frac{v_i^2}{W^2 r_0 R}\right]},$$

where the eigenvalues for ν are $\nu = m + 2n$ as before. The effects of rotation W are much the same here as in the previous section so we specialize to the case of no initial rotation, $W = 0$, and find for the most unstable mode, $m = \nu$,

$$2\omega = -\overline{W}_i(1 - \nu) \pm \sqrt{\overline{W}_i^2(1 - \nu)^2 - 4\nu v_i^2/r_0 R}. \quad (3.28)$$

If finite Larmor radius effects had been omitted, i.e., $\overline{W}_i=0$, the result would have been

$$\omega^2 = -\frac{\nu v_i^2}{r_0 R}\,.$$

Now, however, we can obtain stability if

$$\overline{W}_i^2(1-\nu)^2 > 4\nu v_i^2/r_0 R \text{ or if } \frac{\sqrt{\nu}}{\nu-1}\sqrt{\frac{r_0}{R}} < \frac{\bar{a}_i}{r_0}\,, \quad (3.29)$$

$$\text{and } \nu > 1.$$

For $\nu=0$ $\omega=0, -\overline{W}_i$ which is stable, but for $\nu=1$ $\omega^2 = -v_i^2/r_0 R$ so that this mode is still unstable.

The reason that the mode $\nu=m=1$ is unaffected by these finite Larmor radius corrections is easily understood in terms of the physical model given in Section 1. This mode corresponds to $\Psi = re^{i\varphi}$ so the electric field is a constant independent of position. The physical origin of the stabilizing effect is that the electric field, and hence the electric drift, are different for electrons and ions due to being averaged over the Larmor orbit. For a constant electric field this mechanism is obviously ineffective. However, this mode may perhaps be stabilized by an external conductor. For example, if the parameters are such that by Eq. (3.29) the mode $m=2$ is just stable, then the external conductor at radius $r/r_0=1.7$ will also stabilize $m=1$, as may be seen from the expansion of the Whittaker factors (Eq. 3.20). Since in practice one would of course have to place the conductor completely outside the plasma and as our theory does not apply within a Larmor radius of the surface, it is not clear whether this method of stabilization is feasible.

We have attempted to compare the stability criterion (Eq. 3.29) with the conditions in Coensgen's successful high temperature ion confinement experiment [3]. The data is not sufficiently accurate, nor the theory sufficiently refined to allow a comparison for low m [2, 3] but the stability criterion is certainly well satisfied for higher m values.

Work is in progress on another situation in which strong stabilizing effects are to be anticipated—the surface instabilities [5] such as are encountered in the pinch or stellarator configurations. These are characterized both by low growth rates and by very localized disturbances and may be strongly modified by finite Larmor radius effects.

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Hydromagnetic Instability in a Stellarator*

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When there is a uniform externally imposed longitudinal magnetic field much larger than the field of a discharge current in a cylindrical plasma, one should expect instabilities in the form of a lateral displacement of the plasma column into a helix of large pitch. This problem is examined under the conditions which might occur in the stellarator during ohmic heating. It is shown that the presence of external conductors is unimportant. When effects of the finite length of the machine are considered, a critical current is obtained below which the system is stable to this displacement. Consideration of current distributions other than uniform shows that instabilities which vary as $e^{i(m\theta+kz)}$ can occur for values of m greater than one, so that instabilities can be found for any finite length machine. Experimental results are in agreement with the theory for the $m = 1$ mode. There is no experimental indication of the higher m modes, for which several possible explanations are suggested.

I. INTRODUCTION

KRUSKAL and Tuck[1] (in a paper hereafter referred to as KT) have examined the influence of a longitudinal magnetic field on the instabilities of the pinch effect. The pinch effect is the confinement of a thin column of plasma by means of the magnetic field due to a high-current discharge along the column. Instabilities in the form of lateral "buckling" of the column (in the absence of a longitudinal field) have been predicted theoretically[2] and are well known experimentally.

In KT it was noted that when there is a uniform externally imposed longitudinal field much larger than the field of the discharge current, one should expect instabilities in the form of a lateral displacement of the plasma column into a helix of large pitch. At the wavelength of fastest growth the e-folding time approximates the time it takes a sound wave in the plasma to traverse the radius of the plasma column. In Sec. II we re-examine this problem under the conditions which might be expected to occur in the stellarator during ohmic heating, including the presence of external conductors. In Sec. III we apply this theory to the stellarator and in Sec. IV show that the external conductors are in fact unimportant. In Sec. V we discuss the important effects due to the finite length of the machine, and in Sec. VI the effects of more general current distributions. Finally, in Sec. VII we give the relevant experimental results.

It should be emphasized that the considerations of this paper apply only to stellarators in which the rotational transform[3] results from the large scale geometry of the tube (such as a figure eight shape) rather than from small local perturbation coils (such as helical windings). It is perhaps worth noting that the theoretical results of Secs. II through V are given in less condensed form elsewhere[4]; the appearance of instability and the dependence of the critical current, both on the confining field and on the direction of the plasma current, were predicted in this earlier work well in advance of the experimental confirmation.

II. INFINITE CYLINDER THEORY

We start with the analysis of pinch instability under the conditions considered in KT, but now additionally taking into account the effect of a thin cylindrical sheet conductor coaxial with the plasma. The notations used in KT are redefined for the reader's convenience.

The material pressure, density, and velocity of the plasma are denoted by p, ρ, and $\mathbf{v}$, the magnetic and electric fields by $\mathbf{B}$ and $\mathbf{E}$, the current and charge densities by $\mathbf{j}$ and ϵ, the permeability and permittivity of space by μ_0 and κ_0, and the ratio of specific heats by γ. (We employ mks units throughout.) The equations we use for the interior of the plasma (treated as infinitely conductive) are Eqs. (1) through (8) of KT.

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At an interface between plasma and vacuum, $\mathbf{n}$ denotes the unit normal to the surface directed into the plasma, u the normal velocity of the surface, $\mathbf{j}^*$ and ϵ^* the surface current density and surface charge density, brackets the jump in the enclosed quantity upon crossing the surface from the vacuum into the plasma, a bar under a quantity the arithmetic mean of the values of that quantity on each side of the surface, and p and $\mathbf{v}$ their values in the plasma just inside the surface. The equations we use at the interface are Eqs. (9) through (14) of KT.

Suppose we have a sheet of solid material of small thickness δ fixed in space with vacuum on both sides. Let σ be the volume conductivity of the material and τ a characteristic time for the phenomena to be considered. If δ is much less than the penetration distance $(\tau/\mu_0\sigma)^{\frac{1}{2}}$ of the material, the thickness may be disregarded and the sheet treated as a surface of surface conductivity $\sigma^* = \sigma\delta$. With the same notation as at an interface, our equations at the sheet are then

$$\mathbf{n} \times [\mathbf{B}] = \mu_0 \mathbf{j}^*, \qquad (1)$$

$$\mathbf{n} \cdot [\mathbf{B}] = 0, \qquad (2)$$

$$\mathbf{n} \times [\mathbf{E}] = 0, \qquad (3)$$

$$\mathbf{n} \cdot [\mathbf{E}] = \frac{1}{\kappa_0} \epsilon^* \qquad (4)$$

$$\mathbf{E} - \mathbf{nn}\cdot\mathbf{E} = \frac{\mathbf{j}^*}{\sigma^*}. \qquad (5)$$

We use cylindrical coordinates r, θ, z. Consider the following situation (Fig. 1). Inside the infinite cylinder $r = r_0$ we have a uniform plasma with $p = p_0$, $\rho = \rho_0$, $\mathbf{v} = 0$, $B_r = B_\theta = 0$, $B_z = B_P$, $\mathbf{E} = 0$, $\mathbf{j} = 0$, $\epsilon = 0$. Outside the cylinder $r = r_0$ is a vacuum in which $B_r = 0$, $B_\theta = B_0 r_0/r$, $B_z = B_V$, $\mathbf{E} = 0$. On the cylindrical interface $r = r_0$ we have $j^*_r = 0$, $j^*_\theta = j^*_1$, $j^*_z = j^*_0$, $\epsilon^* = 0$. At $r = r_1 > r_0$ there is a fixed cylindrical thin material sheet of surface conductivity σ^* on which $\mathbf{j}^* = 0$, $\epsilon^* = 0$.

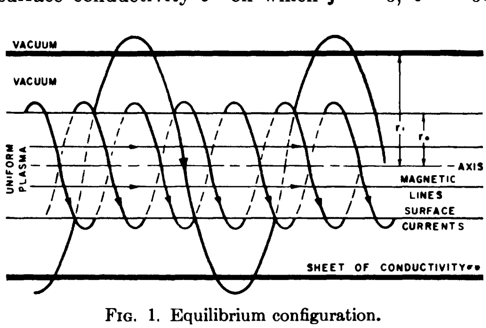

FIG. 1. Equilibrium configuration.

This will be an equilibrium situation (time-independent solution) if the constants r_0, r_1, p_0, ρ_0, B_0, B_P, B_V, j^*_0, j^*_1, and σ^* satisfy

$$B_0 = \mu_0 j_0^*, \qquad B_P - B_V = \mu_0 j_1^*, \qquad (6)$$
$$B_0^2 + B_V^2 - B_P^2 = 2\mu_0 p_0.$$

We now seek solutions of our equations which are close to the equilibrium solution just described. We suppose that every physical quantity is equal to its equilibrium value plus a small perturbation term. We consider all our equations as equations for these perturbation quantities and linearize them in the usual way. We obtain a system of algebraic and differential equations, linear and homogeneous, with r, θ, z, and t as independent variables. The coefficients are obviously independent of θ, z, and t. Any solution of the equations may therefore be obtained as a superposition of elementary solutions, an elementary solution being one in which each dependent variable is a function of r only (or, in the case of sheet quantities, a constant) multiplied by $e^{(im\theta + ikz + \omega t)}$, where m, k, and ω are constants, the characteristic constants of the elementary solution. We may therefore restrict ourselves to a search for the elementary solutions. To make physical sense we must require that m be an integer and that k be real. Without loss of generality we may assume that m is nonnegative.

We next change our notation, each symbol which originally denoted a physical quantity now denoting the coefficient of the exponential in the representation of the perturbation of that quantity. Our equations become linear homogeneous algebraic and ordinary differential equations for these coefficients. We introduce the constants

$$c^2 = \frac{1}{\mu_0\kappa_0}, \qquad s^2 = \frac{\gamma p_0}{\rho_0}, \quad h^2 = \frac{B_P^2}{\mu_0\rho_0},$$

$$\xi^2 = k^2 + \frac{\omega^2}{s^2}, \qquad \eta^2 = k^2 + \frac{\omega^2}{c^2} + \frac{\omega^2}{h^2}, \qquad (7)$$

$$\zeta^2 = k^2 + \frac{\omega^2}{s^2} + \frac{\omega^2}{h^2}, \quad \psi^2 = k^2 + \frac{\omega^2}{c^2};$$

c, s, and h are the velocities of light, sound, and hydromagnetic waves, respectively. The general regular solution of the equations for the plasma is given by Eqs. (19) of KT in terms of an arbitrary constant p_1. (J_m is the mth-order Bessel function of first kind and J'_m its derivative with respect to its argument; J_m and J'_m are here always evaluated for the argument $ir\xi\eta/\zeta$.)

The equations for the vacuum are (4) through (7) of KT with $\mathbf{j} = 0$ and $\epsilon = 0$. For the region $r > r_1$

(outside the fixed conductor) the general regular solution is given by Eqs. (23) of KT in terms of arbitrary constants B_1 and E_1. (H_m is the mth-order Hankel function of first kind and H'_m its derivative with respect to its argument; H_m and H'_m are here always evaluated for the argument $i\psi r$.)

In the general solution for the vacuum region $r_0 < r < r_1$ (between the plasma and the fixed conductor), we have each magnetic and electric field component given as a sum of two expressions, one the same as in (23) of KT except for having B_1 and E_1 replaced by new constants B_2 and E_2, and the other again the same except that B_1 and E_1 are replaced by new constants B_3 and E_3 and at the same time H_m and H'_m are replaced by J_m and J'_m (both evaluated for the argument $i\psi r$).

We now have the solution everywhere expressed in terms of the seven so far arbitrary constants p_1, B_1, E_1, B_2, E_2, B_3, and E_3. We obtain relations between these from the interface and boundary conditions. From conditions (9) through (14) of KT we obtain (only) three independent relations between p_1, B_1, and E_1. From conditions (1) through (5) we obtain four independent relations between all the constants except p_1. Thus we have seven linear homogeneous equations for the seven coefficient constants. The condition that these equations have a nontrivial solution (i.e., that the determinant of their coefficients vanishes) leads to the characteristic equation, which must be satisfied by the characteristic constants of any elementary solution.

We now make the approximation of infinite light velocity by taking $\kappa_0 = 0$. We are interested only in unstable solutions, i.e., solutions for which ω has a positive real part, and we assume that ω is real. It can be proved[5] (at least for σ^* either zero or infinite) that this is no restriction, i.e., that all unstable modes have ω real. Introducing the dimensionless constants and functions

$$\alpha_P = \frac{B_P}{B_0}, \qquad \alpha_V = \frac{B_V}{B_0},$$
$$W = \left(\frac{\rho_0}{2p_0}\right)^{\frac{1}{2}} r_0\omega, \qquad \Sigma = \left(\frac{2p_0}{\rho_0}\right)^{\frac{1}{2}} \mu_0\sigma^*, \tag{8}$$
$$K_m(y) = \frac{J_m(iy)}{iyJ'_m(iy)}, \qquad L_m(y) = \frac{H_m(iy)}{iyH'_m(iy)},$$
$$M_m(y_0, y_1) = \frac{H'_m(iy_1)}{H'_m(iy_0)}\frac{J'_m(iy_0)}{J'_m(iy_1)},$$

the characteristic equation may be written

$$[\alpha_P{}^2y_0{}^2 + (1 + \alpha_V{}^2 - \alpha_P{}^2)W^2]K_m(x)$$
$$= 1 + (\alpha_V y_0 \pm m)^2\Big[L_m(y_0) - \frac{M_m(y_0, y_1)(K_m(y_0) - L_m(y_0))}{1 - M_m(y_0, y_1) + \dfrac{r_0}{r_1}\dfrac{y_1{}^2 + m^2}{\Sigma W}(K_m(y_1) - L_m(y_1))}\Big], \tag{9}$$

where the plus or minus sign is to be chosen according to whether k is positive or negative, and where

$$y_0 = |k|\, r_0, \qquad y_1 = |k|\, r_1,$$
$$x = \left[\left(y_0{}^2 + \frac{2}{\gamma}W^2\right)\frac{\alpha_P{}^2y_0{}^2 + (1+\alpha_V{}^2 - \alpha_P{}^2)W^2}{\alpha_P{}^2y_0{}^2 + \left(1 + \alpha_V{}^2 + \dfrac{2-\gamma}{\gamma}\alpha_P{}^2\right)W^2}\right]^{\frac{1}{2}}. \tag{10}$$

Numerically, $J_m(iy)$ and $J'_m(iy)$ are monotonically increasing functions of y, and $H_m(iy)$ and $H'_m(iy)$ are monotonically decreasing functions of y. Since $y_0 < y_1$ it follows that $0 < M_m(y_0, y_1) < 1$. Also, $K_m(y) > 0$ and $L_m(y) < 0$. We thus see that the second term in the brackets on the right-hand side of (9), which term we shall denote by U, is negative, as is the first term $L_m(y_0)$. The left-hand side of (9) is a monotonically increasing function of W, at least for W very small, for W very large, and for W in the neighborhood of its largest value satisfying (9), and very likely for all W. In any case, it can be proved easily that the largest value of W for which the left-hand side of (9) has a prescribed value is a monotonically nondecreasing function of that prescribed value. Now characteristic equation (9) differs from the corresponding equation for the same equilibrium situation without the conducting sheet at $r = r_1$ [namely, (30) of KT] only in the presence of the term U. It follows, therefore, that the presence of the sheet has quite generally the effect of diminishing the rate of instability.

As was to be expected, $U \to 0$ as $\Sigma \to 0$ or as $r_1 \to \infty$. In the latter case $U \to 0$ very quickly since both $H'_m(iy_1)$ and $1/J'_m(iy_1)$ go to zero exponentially.

As pointed out earlier, Eqs. (1) through (5) are valid if the thickness of the shell is much less than the penetration distance $(\tau/\mu_0\sigma)^{\frac{1}{2}}$ of the shell material, τ being a characteristic time for the phenomena under consideration. With the shell in the form of a cylinder of inner radius r, we can, under rough assumptions determine corresponding equations for the opposite limiting case when $\delta \gg (\tau/\mu_0\sigma)^{\frac{1}{2}}$. We take for granted that $(\tau/\mu_0\sigma)^{\frac{1}{2}} \ll r_1$. We do not know *a priori* the distribution of induced eddy currents in the shell, but we assume, for the sake of having something definite to compute,

[5] Bernstein, Frieman, Kruskal, and Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).

that it is purely in the θ direction. This current distribution turns out to have a characteristic decay time of about $\frac{1}{4}\tau r_1(\tau/\mu_0\sigma)^{-\frac{1}{2}} \gg \tau$. Therefore, there is no appreciable decay during lengths of time of interest, and the shell may be treated as a perfectly conducting sheet, with a radius perhaps exceeding r_1 by something of the order of the penetration distance. Thus, with appropriate values of r_1 and σ^*, Eqs. (1) through (5) may still be considered to hold. The validity of this argument is of course questionable due to the arbitrariness in the choice of the current distribution, but in any case the stabilizing action of the shell must be less than it would be for a perfect conductor, which can be treated as a sheet at radius r_1.

III. APPLICATION

We now wish to apply our theory to the stellarator. One idealization we make is to ignore the curvature of the stellarator and to treat it instead as if it were straightened out to form a right circular cylinder. Since the stellarator has finite length and the present theory deals with an infinite cylinder, it is necessary to impose some periodicity condition. This will be discussed in Sec. V.

Another idealization is to treat the plasma in the stellarator as a uniform plasma with all current on the surface. The effect of modifying this assumption is discussed in Sec. VI.

The equilibrium situation of the theory would seem to represent reasonably well the expected conditions in the stellarator during ohmic heating if we take $B_P = B_V$ to represent the confining field, $2\pi r_0 j_0^*$ to represent the induced plasma current, and the sheet at $r = r_1$ to represent any coaxial cylindrical conductor, such as accelerating or confining field windings or the stainless steel discharge tube. In the stellarator the longitudinal confining field is much larger than the maximum field produced by the plasma current, hence we have $|\alpha| \gg 1$, where $\alpha = \alpha_P = \alpha_V$. It is shown in KT that in this limiting case the only instability is for $m = 1$, to which case we now confine our investigation. It is not hard to show that if we are to obtain a real positive solution W of (9) we must have $y_0 \ll 1$ (i.e., k small), $k\alpha < 0$, W not too large, and $x \ll 1$. Since $K_1(0) = 1$, $L_1(0) = -1$, and (for small y_0 and y_1) $M_1(y_0, y_1) \approx r_0^2/r_1^2$, (9) becomes asymptotically

$$Y^2 + W^2 = 1 - (Y - 1)^2\left(1 + \frac{2}{a^2 - 1 + 2a/\Sigma W}\right), \quad (11)$$

where

$$Y = |\alpha|\, y_0, \qquad a = r_1/r_0. \quad (12)$$

If Σ is finite, Eq. (11) has a positive real solution W only for $0 < Y < 1$; for $Y \to 0$ we find $W \approx 2aY/\Sigma$, while for $Y \to 1$ we find $W \approx [2(1 - Y)]^{\frac{1}{2}}$. If $\Sigma = \infty$, it has a positive real solution only for $a^{-2} < Y < 1$; for $Y \to a^{-2}$ we find $W \approx [2(Y - a^{-2})]^{\frac{1}{2}}$, while for $Y \to 1$ we have $W \approx [2(1 - Y)]^{\frac{1}{2}}$ as before. In any case, the maximum value of W and the value of Y for which it is attained satisfy, in addition to (11), the equation

$$2Y = -2(Y - 1)\left(1 + \frac{2}{a^2 - 1 + 2a/\Sigma W}\right), \quad (13)$$

which is obtained from (11) by partial differentiation with respect to Y. From (11) and (13) we find that

$$Y = 1 - W^2 \quad (14)$$

and that W is determined by

$$W^2\left(1 + \frac{1}{a^2 - 1 + 2a/\Sigma W}\right) = \frac{1}{2}. \quad (15)$$

IV. UNIMPORTANCE OF EXTERNAL CONDUCTOR

In the absence of the conducting sheet (i.e., for $a = \infty$ or $\Sigma = 0$) Eq. (15) gives $W = 1/\sqrt{2}$ or

$$\omega = \frac{1}{r_0}\left(\frac{p_0}{\rho_0}\right)^{\frac{1}{2}} \quad (16)$$

for the maximum rate of instability. In a stellarator we might have a tube of, say, helium plasma of about 2 cm radius. If the ions and electrons were both at temperature T in degrees Kelvin, (16) would become

$$\omega = 3.12 \times 10^3 T^{\frac{1}{2}}\ \mathrm{sec}^{-1}. \quad (17)$$

Since the time scale for operation of the stellarator is of the order of milliseconds, we see that for $T = 10^8$ the instability would grow extremely fast. Even for $T = 10^4$ an instability would be serious if its W were larger than 10^{-3}.

For the conducting sheet to have the effect of reducing the maximum W to a very small value, we see from (15) that it is necessary both for a to be nearly equal to unity and for Σ to be large. Specifically, it is necessary to have

$$a - 1 \leq W^2, \qquad \Sigma \geq W^{-3}. \quad (18)$$

For the stellarator, this means that a conducting shell which is to slow up the instability enough to do any good must in the first place be extremely close to the plasma ($r_1 - r_0 \leq 2 \times 10^{-6}$ cm for

$T = 10^4$). This immediately excludes all conductors except those virtually in contact with the plasma, such as stainless steel tubing. The sheet conductivities of such conductors are unlikely to exceed several hundred mhos by much, which at $T = 10^4$ corresponds roughly to $\Sigma = 4$, whereas Σ would have to be about 10^9 to do any good; at higher temperatures the comparison is even less favorable.

Indeed, it would apparently be hopeless to slow up the instability sufficiently by external conductors even if they were designed for that purpose and it were not necessary to worry about inimical effects they might have on the normal operation of the stellarator. For instance, if one had thick walls of silver ($\sigma = 6 \times 10^7$ mho/meter) arbitrarily close to the plasma, the silver could be treated as a perfectly conducting sheet at a radius greater than the plasma radius r_0 by approximately the penetration distance of the silver, and computation shows that, even for a plasma temperature as low as 0.3°K, the instability would then *e*-fold in a millisecond.

V. PERIODICITY CONDITION

Now that we have seen that external conductors have a negligible effect on the instability, we turn to an examination of the restrictions on perturbations imposed by the geometry of the stellarator. We wish to treat the stellarator tube as if it were straightened out to a right circular cylinder. Put another way, we wish to define coordinates r, θ, z in the tube which locally are approximately cylindrical coordinates and in terms of which the (inner) surface of the tube is approximately the surface $r = r_0$. It is natural to choose the curve $r = 0$ to be the magnetic axis of the stellarator (i.e., the magnetic line of force which closes upon itself after one traversal of the length of the tube); r, θ to be polar coordinates in each cross section of the tube; z to be constant on each cross section; and z to be arc-length along the curve $r = 0$ (with the sign of dz chosen so as to make r, θ, z a right-handed coordinate system). It remains only to determine the relative rotation of the polar coordinates in different cross sections, i.e., to determine the direction $\theta = 0$ (say) in each cross section. Choosing an arbitrary vector at $r = 0$ lying in one cross section to give the direction $\theta = 0$ there, we consider a parallel vector at the point $r = 0$ of a neighboring cross section. This parallel vector does not in general lie in the neighboring cross section, but we may choose its projection thereon as the direction $\theta = 0$. In this way the direction $\theta = 0$ may be carried successively around the length of the stellarator. That this is the natural method of relating the values of θ in different cross sections may be seen in several ways. One way is to observe that what we have done is equivalent to requiring that the coordinates r and θ be invariant when the z cross section is projected onto the $z + dz$ cross section in the direction of the magnetic axis; the resultant values of r and θ in the $z + dz$ cross section do not exactly constitute polar coordinates, but the deviation is of the order dz^2 and is therefore negligible. Another way is to observe that the lines of force of the confining magnetic field approximate to curves of constant r and θ.

The "cylindrical" coordinates we have defined in the tube are not single-valued functions of position (except for r, which is the distance from the magnetic axis). If we follow the values of θ and z along a closed curve which goes once around the length of the stellarator in the direction of positive dz, we find that when we have returned to the starting point, z has increased by the length L of the magnetic axis and θ has increased by a definite angle ι depending only on the geometry of the stellarator (and not at all on the starting point or the particular curve chosen). This is called the rotational transform angle.

It can be shown by standard methods of the differential geometry of space curves that $-\iota$ is equal to the integral of the torsion of the magnetic axis with respect to its arc length, once around the stellarator. (The torsion of a curve is the negative of the rate of rotation, with respect to arc length, of the osculating plane, i.e., of the plane determined by the tangent and the radius of curvature. The positive direction of rotation is determined by the right-hand rule from the direction along the curve in which the arc length is taken as increasing.) For stellarators of twisted figure eight shape[4], let ϕ be the angle through which each end of a plane figure eight must be rotated to arrive at that shape, the positive direction of rotation for each end being clockwise as seen from beyond that end. It is then easily seen that

$$\iota = -4\phi. \tag{19}$$

Now, (r, θ, z) and $(r, \theta + \iota, z + L)$ represent the same point in the tube, so in our perturbation theory we can allow only elementary solutions for which $m\iota + kL$ is an integral multiple of 2π. We recall that the unstable perturbations we are concerned with have $m = 1$, $k\alpha < 0$, $0 < Y < 1$. Since $Y = |\alpha k| r_0$, we see that there is one allowable perturbation for each integer h (positive, negative, or zero) satisfying

$$0 < \alpha(r_0/L)(\iota + 2\pi h) < 1. \tag{20}$$

Thus the condition that no instability be allowable is that

$$\alpha(\iota + 2\pi h) \geq L/r_0, \tag{21}$$

where h is that integer which gives the left-hand side of (21) its smallest positive value.

It is clear that the stability criterion (21) depends not only on the magnitude of α but also upon its sign, unless ι happens to be an integral multiple of π. We note that α is positive or negative accordingly as the induced longitudinal plasma heating current has the same or the opposite direction as the longitudinal confining magnetic field.

Condition (21) is more conveniently expressed, for application, in terms of the plasma current $I = 2\pi r_0 j_0^* = 2\pi r_0 B_0/\mu_0$. Since $\alpha = B_V/B_0$, (21) may be written

$$(B_V/I)(\iota + 2\pi h) \geq \mu_0 L/2\pi r_0^2. \tag{22}$$

VI. CURRENT DISTRIBUTION EFFECTS

Some longitudinal current distributions more general than the purely surface current case of Sec. II are treated elsewhere[6] by means of the energy principle.[5] We quote the results without the complications of an external conductor. The fluid pressure p is taken to be zero, B_z is again taken to be much larger than B_θ, and the condition for cylindrically symmetric equilibrium,

$$\frac{1}{2}\frac{\partial}{\partial r}(B_\theta^2 + B_z^2) + B_\theta^2/r = 0, \tag{23}$$

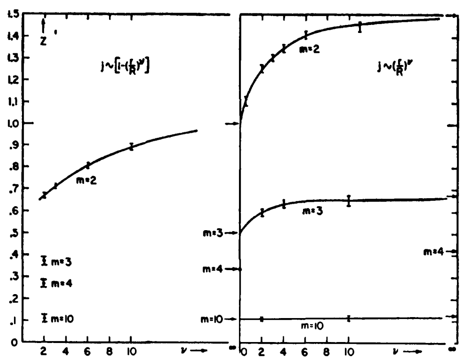

FIG. 2. Limit of range of instability for axial current distributions $j \sim [1 - (r/r_0)^\nu]$ and $j \sim (r/r_0)^\nu$.

[6] Johnson, Oberman, Kulsrud, and Frieman, U. S. Atomic Energy Commission Report No. NYO-7904 (PM-S-34) (1958).

is satisfied for B_θ any function of r by choosing B_z to be the appropriate nearly constant function. The conclusion then is that there is one allowable mode of instability for each (positive or negative) integer h satisfying

$$1/Z < \alpha \frac{r_0}{L}(m\iota + 2\pi h) < m, \tag{24}$$

where $Z > 0$ depends on the function B_θ, i.e., on the distribution of current j_z, and on m. This is the generalization of (20).

For $m = 0$, there is clearly no instability. For $m = 1$, Z becomes infinite independently of the current distribution, and thus (24) reduces to (20).

For $m > 1$, Z is given as a function of a positive exponent ν in Fig. 2 for j_z proportional to $1 - (r/r_0)^\nu$ and also to $(r/r_0)^\nu$. The first type of distribution with $\nu = \infty$ and the second with $\nu = 0$ are identical, both representing uniform current, and have $Z = 1/(m - 1)$.

For both types of distribution Z is greater than $1/m$ and increases monotonically with ν. We see that for each $m > 1$ there are thus always ranges of values of α for which there is instability. These ranges increase with ν.

As $\nu \to \infty$ the second type of distribution approaches the sheet current case considered earlier [and Z approaches $(m + 1)/(m - 1)m$]. The nonvanishing of the ranges of instability in the limit seems somewhat paradoxical because the limiting sheet current case is stable for $m > 1$. The resolution is probably that the unstable perturbations become stabilized by nonlinear effects at smaller and smaller amplitudes as one goes to the limit.

These results have applied when the plasma is surrounded by vacuum. If instead it is surrounded by pressureless plasma and the equilibrium fields are the same, then the results for $m = 0$ and $m = 1$ are the same, but for $m \geq 2$ there is now complete stability. It seems uncertain whether vacuum or pressureless plasma is the better approximation to conditions in the region between the main plasma and the tube wall in a stellarator.

VII. EXPERIMENTAL RESULTS

It has been shown in the previous sections that a figure-eight shaped stellarator should exhibit an $m = 1$ instability for currents greater than a critical value determined by (22). The magnitude of this critical current depends on the geometry of the system (through ι, the rotational transform angle, and L the axial length), the plasma cross-sectional area (through r_0 the radius), the magnitude of the longitudinal confining magnetic field B_V, and on whether the current direction is along or opposite to the magnetic field. It is interesting to note that the critical current in either of the two directions is just sufficient to cause a resultant rotation of either zero or 2π in the magnetic lines of force just outside the plasma (i.e., to make the lines of force close on themselves once around the machine). This result is independent of the form of current distribution.

The quantities ι, I, B_V, and L are all easily measurable. If all the lines of force are exactly parallel to the walls of the discharge tube, then r_0 is simply the discharge tube radius. If this is not the case, r_0 is then the radius of the innermost magnetic surface which anywhere touches the discharge tube walls. By means of a collimated electron beam, it is experimentally possible to determine r_0 under low-field, steady-state conditions to an accuracy of about 10%. These values are confirmed by measurements of the plasma inductance. In various stellarators the effective radius (called the radius of the aperture) is from 50% to 90% of the discharge tube radius. A summary of some pertinent

TABLE I. Physical properties of stellarators.

Stellarator model	ι	L	Tube inside radius	Aperture radius	Critical current at 10 kilogauss
B-1	196°	450 cm	2.2 cm	1.6 cm	810 or 970 amp
B-2	196°	600	2.2	1.6	610 or 730
B-3	196°	600	2.2	2.0	950 or 1140

quantities is given in Table I, where the critical current is calculated from (22), which becomes on solving for I

$$I = \frac{B_V 2\pi r_0^2}{\mu_0 L}\,\eta, \qquad (25)$$

with $\eta = 0.911\pi$ and 1.089π for the two possible directions of current and B_V, r, L, and μ_0 are all in mks units.

The observational results to be expected as a consequence of this instability are not entirely obvious. The observations on the B-1 stellarator where the effects seem particularly apparent will first be described in some detail, followed by a summary of similar evidence from other devices

The ohmic heating electric field is usually applied in the form roughly of a square wave of adjustable amplitude by means of a transformer which links the discharge tube. A more complete description of

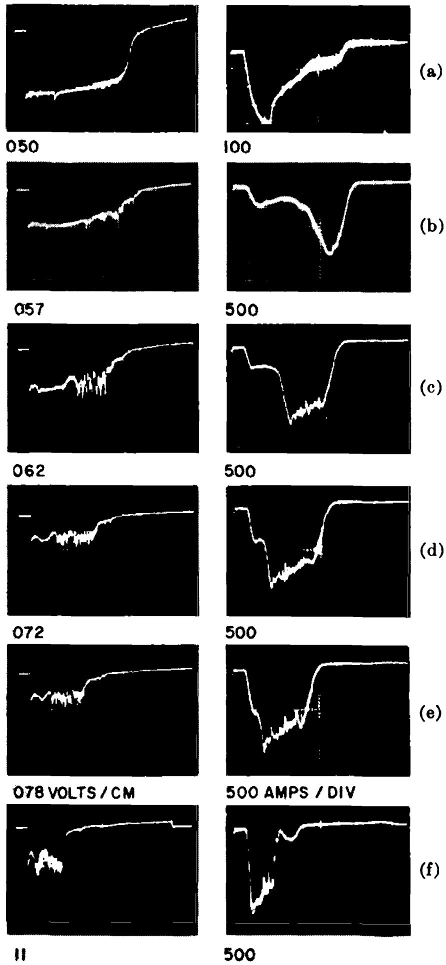

FIG. 3. Plasma current and applied electric fields as functions of time in B-1.

the characteristics of the stellarator is given by Coor *et al.*[7] The duration time is self-limited by saturation of the transformer. The plasma current and applied electric fields are displayed as functions of time on oscilloscopes. Figure 3 shows these data for six different E fields, applied to a helium discharge at an initial pressure of 6×10^{-4} mm Hg. The oscilloscope sweep speed is 1 millisecond per cm from left to right and the initially applied electric field in volts per cm is given in each case. The field

[7] Coor, Cunningham, Ellis, Heald, and Kranz, Phys. Fluids 1, 411 (1958).

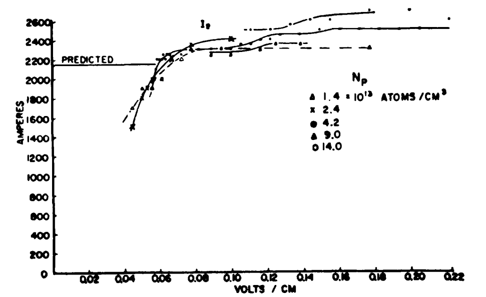

FIG. 4. Peak plasma current as a function of applied electric field at various pressures in B-1.

gradually falls (because of the partial discharge of a capacitor bank), and fluctuations appear which are the result of plasma inductance effects. As successively higher electric fields are applied the current rises more rapidly and (in the first four cases) to a higher peak value. However, for fields above about 0.06 v/cm there is very little dependence of peak current on applied voltage as shown in Fig. 4, which shows peak current plotted against applied electric field at various gas pressures. The current essentially reaches a plateau, the level of which is roughly independent of the pressure. As shown in the figure the plateau value of current agrees with that predicted from (25). Similar sets of data taken at other values of confining field produce similar effects at current levels proportional to the magnetic field in quantitative agreement with prediction.

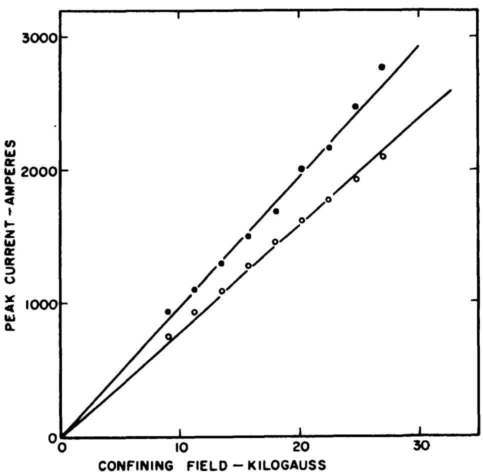

FIG. 5. Peak plasma current as a function of confining field in B-1. The two curves are for plasma current in opposite directions.

These are the solid circles in Fig. 5. However, the most striking effect is that of reversing the direction of the current with respect to the magnetic field. These data are plotted as open circles in Fig. 5. The ratio of slopes of the two lines is 1.22. On the basis of the twist angle of B-1, one would expect a ratio of 1.19. This difference is well within experimental error.

Another point of interest is that whenever the current rises above this critical limiting value, the current and voltage become quite noisy as may be seen in Fig. 3, and large amounts of impurities appear in the discharge.[7]

Further verification is offered by the fact that the critical value of current is the same in a hydrogen discharge as in a helium discharge.

In all cases it is possible to drive the current well above the critical current if a high enough electric field is applied. Figure 6 shows, for example, a plot of peak current *vs* ohmic heating field for 3 different values of magnetic confining field in the B-2 stellarator. There is once more a definite leveling off at a current consistent with prediction, but for high ohmic heating fields the current does continue to rise. However, in this case a slight step or irregularity appears at approximately the critical current (shown by the dotted lines in Fig. 6).

In the case of B-3, much more care was taken in alignment of the field coils and the discharge tube. As a result the aperture area as measured by the electron gun is 13 cm^2 as compared with about 8 cm^2 in both B-1 and B-2. Correspondingly higher plateau currents are expected and are observed in this case up to a magnetic confining field of 53 000 gauss. The discharge tubes in all the devices previously mentioned are stainless steel with bakable vacuum systems which may be pumped down to pressures of the order of 10^{-10} mm Hg.

B-1 was formerly operated with relatively "dirty" walls of stainless steel and later of Pyrex, such that only base pressures of about 10^{-5} mm Hg were possible. The metal tube required about 4 times as much electric field to get initial breakdown and showed very little evidence of the current leveling off. In both these respects, on the other hand, the "dirty" glass system was quite similar to the "clean" metal system.

Clearly there is very satisfactory agreement between theory and experiment with regard to the $m = 1$ mode of instability. However, there is no experimental indication of the existence of the higher m modes. One possible explanation of this is that the region between the main body of plasma

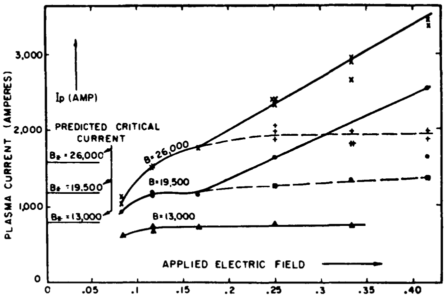

FIG. 6. Peak plasma current as a function of applied electric field at various confining fields in B-2.

and the tube wall might contain enough charged particles to act as a good conductor, i.e., as a pressureless plasma, in which case the higher m modes would be stable. It is of course quite possible that higher m modes of instability are present but produce effects less easily observable than those due to the $m = 1$ mode. For instance, they might merely distort the original equilibrium configuration into a new not greatly shifted stable configuration. This would be consistent with the proposed resolution of the paradox in Sec. VI. It seems highly plausible that for long wavelength modes with $m \neq 1$, nonlinear terms become significant when the displacement becomes comparable with the plasma radius, whereas for long wavelength $m = 1$ modes, the linearized perturbation theory remains valid until the displacement becomes comparable with the wavelength.

The Influence of an Axial Magnetic Field on the Stability of a Constricted Gas Discharge

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Abstract. It is shown that a suitable arrangement of axial magnetic fields and conducting walls will stabilize a constricted gas discharge if all the discharge current flows on its surface. For stability the axial field in the discharge exceeds that in the low density region surrounding it and the wall radius cannot exceed the discharge radius by more than a factor of five. In the other extreme case, when the axial current is uniformly distributed across the discharge and the internal field cannot exceed the external field, no complete stability results.

The problem of the stability of general current and field configurations is reduced to the solution of two first order differential equations and a dispersion relation. These equations are discussed qualitatively and we obtain a generalization of a sufficiency condition for stability in the absence of an axial field, which was given in a previous paper.

§ 1. Introduction

A CONSTRICTED gas discharge in an insulating tube is observed to be highly unstable; it wriggles about and reaches the tube walls. Recently published photographs of this behaviour include those of Carruthers and Davenport (1957). Some account of the theory of these instabilities has been given by Kruskal and Schwarzschild (1954) and Tayler (1957 a, to be referred to as I). If a high current constricted discharge is to be obtained in a quasi-stationary state as discussed by Pease (1957), it is necessary to remove these instabilities. The use of external magnetic fields of two types is immediately suggested: (*a*) an axial magnetic field which may, and in general will, be present in both the discharge and the low-density region surrounding it, and (*b*) fields due to eddy currents in highly conducting tube walls. We shall discuss these two possibilities in this paper though they are, of course, not exhaustive. First considerations show that image currents act best on long wavelength instabilities and axial fields on those of short wavelength; can a suitable combination stabilize all wavelengths?

In this paper we are mainly interested in the situation in which a uniform axial magnetic field is initially present in a cylindrical tube containing a slightly conducting gas. At this stage an axial electric field is introduced and the discharge is set up. Further development depends on how rapidly power can be introduced into the system. If the rise in conductivity of the plasma is sufficiently rapid, the axial magnetic field will be largely trapped within the plasma as it contracts. Thus we may hope to reach a quasi-stationary situation, when the discharge is constricted, in which both the axial and azimuthal currents are carried in a thin layer near the discharge surface and there is a virtual separation of axial and azimuthal fields. If the conductivity rises less rapidly, the currents and fields will be more uniformly distributed in the equilibrium configuration. In either case subsequent development due to dissipative processes, because of the finite though large conductivity of both plasma and walls, will lead to interpenetration of the fields. We are thus interested in the stability properties of fairly general configurations of both axial and azimuthal fields.

What we in fact show is that it is possible to obtain completely stable configurations of the first type; that is those in which the currents flow on the discharge surface and the axial magnetic field is mainly trapped within the plasma. These results depend critically on two factors. Stability cannot be obtained without the presence of conducting walls and these walls must be relatively close to the discharge; in the most favourable case the cylindrical wall cannot have a radius greater than five times the discharge radius. Secondly if the discharge is to be noticeably constricted a very high degree of trapping, of the axial field within the plasma, is required; the stable wall position rapidly approaches the discharge as the ratio of internal to external axial field is reduced. We have also been able to consider another extreme case in detail. If the axial current is uniformly distributed across the discharge and the axial field is also uniform, no completely stable configurations exist.

Some of the results contained in this paper have been obtained independently in Russia and reported by Artsimovitch in a talk given at Stockholm in September 1956†. Problems similar in some respects to those considered here have been studied by Dungey and Loughhead (1954) and Roberts (1956) amongst others; although their analysis is similar their emphasis is on properties of magnetic fields rather than current channels.

The remainder of this paper is arranged as follows. In the next section the fundamental equations and assumptions are stated. In § 3 we discuss the properties of completely stable configurations in which all currents are carried on the discharge surface. In § 4 we formulate the equations for general current distributions, and in § 5 obtain a full solution for one special case. In the final section we discuss our results and further problems.

§ 2. Assumptions and Basic Equations

In what follows we shall make the following assumptions:

(i) After the initial contraction of the discharge a quasi-equilibrium state is reached. Thus we ignore any instabilities which may occur in this collapse stage.

(ii) The conductivity of the discharge in the quasi-equilibrium state is high enough to be regarded as infinite in stability calculations. This means that the time scale of instabilities which we studied in I is much shorter than the time scale of field penetration due to dissipation. The validity of this assumption can be checked by comparing the times r_0/c_s and $4\pi\sigma r_0^2/c^2$ where r_0 is the discharge radius, c_s the velocity of sound, σ the discharge conductivity and c the velocity of light.

(iii) The wall conductivity may be treated as infinite and we ignore the possible presence of gaps in the conducting walls.

(iv) We consider only equilibrium configurations in infinite cylindrical geometry and with axial symmetry.

† I am informed that similar results have been obtained in the United States by M. Kruskal and J. L. Tuck and by M. Rosenbluth and are to be published.

The method of normal mode analysis used in this paper has already been described in I. Perturbations about the equilibrium state are such that any variable takes the form

$$q = q_0 + q_1 \exp\{i(m\theta + kz) + \omega t\} \qquad \ldots\ldots(2.1)$$

where k is a real wave number and m is an integer positive or negative. In I all equations were even in m and k so that they could both be taken as positive; in the present work it is found that m, k can both be taken as positive provided that we consider positive and negative values of the axial field. The equilibrium configurations of I are altered by the introduction of axial magnetic fields, and perfectly conducting walls at $r = R_0$. As before displacement currents are neglected and we again state without proof that overstability cannot occur.

Plasma equations.

The plasma equations, restated for reference purposes, are as follows:

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} p + \frac{\mathbf{j} \times \mathbf{B}}{c} \qquad \ldots\ldots(2.2)$$

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \rho \mathbf{v} \qquad \ldots\ldots(2.3)$$

$$\operatorname{curl} \mathbf{B} = \frac{4\pi \mathbf{j}}{c} \qquad \ldots\ldots(2.4)$$

$$\operatorname{div} \mathbf{B} = 0 \qquad \ldots\ldots(2.5)$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \qquad \ldots\ldots(2.6)$$

$$\operatorname{div} \mathbf{E} = 4\pi\epsilon \qquad \ldots\ldots(2.7)$$

$$\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} = 0 \qquad \ldots\ldots(2.8)$$

and either

$$\frac{1}{p}\frac{dp}{dt} = \frac{\gamma}{\rho}\frac{d\rho}{dt} \qquad \ldots\ldots(2.9)$$

or

$$\operatorname{div} \mathbf{v} = 0. \qquad \ldots\ldots(2.10)$$

Vacuum equations.

In the vacuum the perturbed magnetic field $\mathbf{B}_1$ is both irrotational and solenoidal, leading to

$$B_{1r}{}^v = kC_1 K_m'(kr) + k C_2 I_m'(kr) \qquad \ldots\ldots(2.11)$$

$$B_{1\theta}{}^v = \frac{im}{r} C_1 K_m(kr) + \frac{im}{r} C_2 I_m(kr) \qquad \ldots\ldots(2.12)$$

$$B_{1z}{}^v = ik\, C_1 K_m(kr) + ik\, C_2 I_m(kr) \qquad \ldots\ldots(2.13)$$

where C_1 and C_2 are constants, arbitrary at present.

Boundary conditions.

The boundary conditions have been stated fully in I. Here it is sufficient to state that for the purpose of obtaining the dispersion relation we need to apply the following two conditions on the perturbed plasma surface:

$$P = p + \mathbf{B}^2/8\pi \text{ continuous,} \qquad \ldots\ldots(2.14)$$

$$B_n \text{ continuous.} \qquad \ldots\ldots(2.15)$$

Only the second condition is required on the rigid conducting wall where it gives

$$\frac{C_1}{C_2} = -\frac{I_m'(kR_0)}{K_m'(kR_0)}. \qquad \ldots\ldots(2.16)$$

The equation of the perturbed plasma surface is

$$r = r_0 + r_1 \exp\{i(m\theta + kz) + \omega t\}$$

and the unit normal vector to this surface is

$$\mathbf{n} = (-1, 0, 0) + \left(0, \frac{imr_1}{r_0}, ikr_1\right)\exp\{i(m\theta + kz) + \omega t\} \ldots\ldots(2.17)$$

§ 3. Stable Configurations with only Surface Currents

In equilibrium we have an ideally conducting cylindrical plasma of radius r_0, surrounded by a vacuum, which is itself enclosed by an ideally conducting cylindrical wall at radius R_0. The plasma has uniform density ρ_0, pressure p_0 and ratio of specific heats γ. It carries a magnetic field $(0, 0, B_{\theta 0} b_i)$ and on its surface there is a sheet current $(0, j_{\theta 0}^*, j_{z0}^*)$. In the vacuum there is a magnetic field $(0, B_{\theta 0} r_0/r, B_{\theta 0} b_e)$. The axial fields inside and outside the plasma are therefore expressed as fractions b_i and b_e respectively of the azimuthal field $B_{\theta 0}$ at the plasma surface.

The configuration is in equilibrium provided that

$$\left.\begin{aligned} j_{\theta 0}^* &= \frac{cB_{\theta 0}}{4\pi}[b_i - b_e] \\ j_{z0}^* &= \frac{cB_{\theta 0}}{4\pi} \\ \text{and} \qquad p_0 &= \frac{B_{\theta 0}^2}{8\pi}[1 + b_e^2 - b_i^2]. \end{aligned}\right\} \qquad \ldots\ldots(3.1)$$

We consider perturbations of this equilibrium configuration of the type given by equation (2.1). Equations (2.6) and (2.8) combine to give

$$\mathbf{B}_1 = \frac{ikB_{\theta 0} b_i}{\omega}\left[v_{1r}, v_{1\theta}, v_{1z} + \frac{i}{k}\operatorname{div}\mathbf{v}_1\right]. \qquad \ldots\ldots(3.2)$$

Equations (2.3) and (2.9) give

$$p_1 = -\frac{\rho_0 c_s^2}{\omega}\operatorname{div}\mathbf{v}_1, \qquad \ldots\ldots(3.3)$$

where $c_s^2 = \gamma p_0/\rho_0$, so that c_s is the velocity of sound in the undisturbed plasma.

The third component of equation (2.2) gives

$$v_{1z} = -\frac{ikp_1}{\rho_0\omega} = \frac{ikc_s^2}{\omega^2}\operatorname{div}\mathbf{v}_1, \qquad \ldots\ldots(3.4)$$

so that equation (3.2) becomes

$$\mathbf{B}_1 = \frac{ikB_{\theta 0} b_i}{\omega}\left[v_{1r}, v_{1\theta}, v_{1z}\left(1 + \frac{\omega^2}{k^2 c_s^2}\right)\right]. \qquad \ldots\ldots(3.5)$$

Then using equations (3.3) and (3.5) in conjunction with the first two components of equation (2.2) we obtain expressions for v_{1r} and $v_{1\theta}$ in terms of v_{1z}:

$$v_{1r} = -\frac{iDv_{1z}[k^2 + \omega^2/c_H^2 + \omega^2/c_s^2]}{k[k^2 + \omega^2/c_H^2]}, \qquad \ldots\ldots(3.6)$$

$$v_{1\theta} = \frac{mv_{1z}[k^2 + \omega^2/c_H^2 + \omega^2/c_s^2]}{kr[k^2 + \omega^2/c_H^2]}, \qquad \ldots\ldots(3.7)$$

where D stands for d/dr and $c_H{}^2 = B_{\theta 0}{}^2 b_i{}^2/4\pi\rho_0$, so that c_H is the hydromagnetic velocity in the undisturbed plasma. Using equations (3.4), (3.6) and (3.7) we now obtain one equation for v_{1z}:

$$rD[rDv_{1z}] = v_{1z}\left[m^2 + \frac{(k^2+\omega^2/c_H{}^2)(k^2+\omega^2/c_s{}^2)}{[k^2+\omega^2/c_H{}^2+\omega^2/c_s{}^2]}r^2\right]. \qquad \ldots\ldots(3.8)$$

This is a modified Bessel equation and its solution is

$$\left.\begin{aligned} v_{1z} &= CI_m(\alpha r) \\ v_{1\theta} &= \frac{m[k^2+\omega^2/c_H{}^2+\omega^2/c_s{}^2]}{k[k^2+\omega^2/c_H{}^2]}C\frac{I_m(\alpha r)}{r} \\ v_{1r} &= \frac{-i\alpha[k^2+\omega^2/c_H{}^2+\omega^2/c_s{}^2]}{k[k^2+\omega^2/c_H{}^2]}CI_m{}'(\alpha r) \end{aligned}\right\} \qquad \ldots\ldots(3.9)$$

where the expressions for $v_{1\theta}$ and v_{1r} have been obtained from equations (3.7) and (3.6), C is an arbitrary constant and

$$\alpha^2 = \frac{(k^2+\omega^2/c_H{}^2)(k^2+\omega^2/c_s{}^2)}{[k^2+\omega^2/c_H{}^2+\omega^2/c_s{}^2]}.$$

The boundary conditions can now be applied on the perturbed plasma boundary, using the vacuum solutions of § 2, and we obtain

$$\alpha k r_0{}^2\left[b_i{}^2 + \frac{\gamma(\omega^2/c_s{}^2)(1+b_e{}^2-b_i{}^2)}{2(k^2+\omega^2/c_s{}^2)}\right]\frac{I_m(\alpha r_0)}{I_m{}'(\alpha r_0)}$$
$$= kr_0 + (m+b_e kr_0)^2\left[\frac{K_m(kr_0)I_m{}'(kR_0) - I_m(kr_0)K_m{}'(kR_0)}{K_m{}'(kr_0)I_m{}'(kR_0) - I_m{}'(kr_0)K_m{}'(kR_0)}\right] \qquad \ldots\ldots(3.10)$$

It is convenient to express this equation in terms of non-dimensional variables. Thus we make the substitutions

$$kr_0 = X_0,\quad \alpha r_0 = U_0,\quad \omega r_0/c_s = W_0 \quad\text{and}\quad R_0/r_0 = \Lambda. \qquad \ldots\ldots(3.11)$$

Then equation (3.10) becomes

$$X_0U_0\left[b_i{}^2 + \frac{\gamma W_0{}^2(1+b_e{}^2-b_i{}^2)}{2(X_0{}^2+W_0{}^2)}\right]\frac{I_m(U_0)}{I_m{}'(U_0)}$$
$$= X_0 + (m+b_eX_0)^2\left[\frac{K_m(X_0)I_m{}'(\Lambda X_0) - I_m(X_0)K_m{}'(\Lambda X_0)}{K_m{}'(X_0)I_m{}'(\Lambda X_0) - I_m{}'(X_0)K_m{}'(\Lambda X_0)}\right]. \qquad \ldots\ldots(3.12)$$

In stable configurations, it must be possible to choose values of the three parameters b_i, b_e and Λ so that for no positive values of X_0 and for no positive integral values of m can equation (3.12) have a positive root for $W_0{}^2$. As mentioned in § 2, in order to allow for all possible perturbations we must consider positive and negative values of b_i and b_e. In fact equation (3.12) is even in b_i, and we shall find that we need only consider negative b_e as positive b_e always gives greater stability. b_i, b_e and Λ are not completely free parameters as they must satisfy the inequalities

$$1+b_e{}^2 \geqslant b_i{}^2,\quad \Lambda \geqslant 1. \qquad \ldots\ldots(3.13)$$

The first inequality follows from the third of equations (3.1) since the plasma pressure must be positive.

The right-hand side of equation (3.12) is independent of W_0; the left-hand side varies between $b_i{}^2X_0{}^2I_m(X_0)/I_m{}'(X_0)$ and infinity as $W_0{}^2$ varies between zero and infinity and it is never less than $b_i{}^2X_0{}^2I_m(X_0)/I_m{}'(X_0)$. Thus equation (3.12) cannot have a solution for positive W_0^2 if

$$X_0 + (m + b_e X_0)^2 \left[\frac{K_m(X_0) I_m'(\Lambda X_0) - I_m(X_0) K_m'(\Lambda X_0)}{K_m'(X_0) I_m'(\Lambda X_0) - I_m'(X_0) K_m'(\Lambda X_0)} \right] < \frac{b_i^2 X_0^2 I_m(X_0)}{I_m'(X_0)} . \qquad \ldots\ldots(3.14)$$

This is both a necessary and sufficient condition for stability, so that if inequality (3.14) fails to be satisfied for any m, X_0 the configuration is unstable. The inequality has two obvious properties. There cannot be complete stability in the absence of a trapped axial field ($b_i = 0$); for then the right-hand side of inequality (3.14) is zero while the left-hand side is certainly positive for negative b_e and m and X_0 satisfying $m + b_e X_0 = 0$. Secondly if the inequality is satisfied for any negative b_e it is also satisfied for the corresponding positive b_e; the additional term

$$4 m b_e X_0 \left[\frac{K_m(X_0) I_m'(\Lambda X_0) - I_m(X_0) K_m'(\Lambda X_0)}{K_m'(X_0) I_m'(\Lambda X_0) - I_m'(X_0) K_m'(\Lambda X_0)} \right]$$

is negative. Thus as mentioned above we need only consider negative b_e. The instability, in the absence of a trapped field, with the wave number satisfying $m + b_e X_0 = 0$, occurs when the perturbation helix and the equilibrium magnetic field helix at the plasma surface coincide.

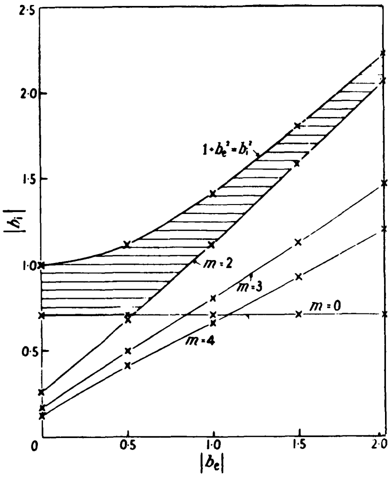

Figure 1. Stability diagram for surface current, no conducting walls. Stability is obtained for values of $|b_e|$, $|b_i|$ lying above the curve corresponding to a given value of m and below the curve $1 + b_e^2 = b_i^2$. Complete stability for $m \neq 1$ is obtained in the shaded region.

We consider the behaviour of the inequality (3.14) first taking into account axial magnetic fields but no conducting walls ($\Lambda = \infty$). In this case we find it is possible to stabilize both the $m = 0$ modes and those with $m > 1$ for suitable values of b_i and b_e, but that without conducting walls it is not possible to stabilize the $m = 1$ modes. For given m and $|b_e|$ there is a critical value of $|b_i|$ above which complete stability is obtained; at the same time $|b_i|$ must satisfy inequality (3.13). The results are shown in table 1 and also graphically in figure 1. In the absence of conducting walls the critical value of $|b_i|$ for $m = 0$ is independent of $|b_e|$. It can be seen that the curve $1 + b_e^2 = b_i^2$ and the critical points for $m = 0$ and $m = 2$

define a region (shaded in figure 1) in which all but the $m = 1$ modes are stable. In this region $m = 1$ disturbances of high wave number will be stable. In table 2 we show the critical wave numbers above which stability occurs for a series of points on the upper and lower boundaries of the shaded region in figure 1.

Table 1. Surface Current. Critical Values of $|b_i|$ for Stability

m \ $\|b_e\|$	0·0	0·5	1·0	1·5	2·0
0	0·707	0·707	0·707	0·707	0·707
2	0·259	0·676	1·122	1·592	2·073
3	0·169	0·492	0·801	1·130	1·468
4	0·126	0·409	0·660	0·926	1·202

Table 2. Surface Current. Critical Wave Numbers for Stability, $m = 1$

$\|b_e\|$	0·0	0·5	1·0	1·5	2·0
$\|b_i\|$	1·000	1·118	1·414	1·803	2·236
X_0	0·450	0·869	0·736	0·571	0·455
$\|b_i\|$	0·707	0·707	1·122	1·592	2·073
X_0	1·025	1·886	0·991	0·661	0·497

If conducting walls are now introduced we find that the $m = 1$ modes can also be completely stabilized, though only if the walls are quite close to the discharge. The values of $|b_i|$, $|b_e|$ for which complete stability can be obtained and the corresponding values of Λ are shown in tables 3 and 4 and figure 2. Table 3 shows the values of Λ below which complete stability for $m = 1$ can be obtained if $|b_i|$ has its maximum value $(1 + b_e^2 = b_i^2)$. It can be seen that for $b_e = 0$, Λ has a maximum value of only 5, and that as $|b_e|$ increases from zero the value of Λ rapidly

Table 3. Surface Current. Critical Wall Radii for Stability, $m = 1$

$\|b_e\|$	0·0	0·1	0·2	0·5	1·0
$\|b_i\|$	1·000	1·005	1·020	1·118	1·414
Λ	5·00	3·31	2·62	1·81	1·39

decreases. For given $|b_e|$ as $|b_i|$ decreases the critical value of Λ decreases. Λ must be greater than 1 and the minimum values of $|b_i|$ for which stability is obtained are shown in table 4. For other values of Λ between 1 and 5 we can obtain similar stability curves in the $(|b_i|, |b_e|)$ plane and several of these are shown in figure 2.

Table 4. Surface Current. Minimum Values of $|b_i|$ for Stability, $m = 1, \Lambda = 1$

$\|b_e\|$	0·0	0·2	0·5	1·0	1·5	2·0
$\|b_i\|$	0·000	0·429	0·683	1·114	1·580	2·061

The region of complete stability is mainly determined by the $m = 1$ behaviour but the $m = 0$ disturbances must also be considered. With finite Λ stability with $m = 0$ is obtained for $|b_i|$ greater than the value given by

$$b_i^2 = \tfrac{1}{2} - b_e^2/(\Lambda^2 - 1). \qquad \ldots\ldots(3.15)$$

Solutions of equation (3.15) for several values of Λ, $|b_e|$ are given in table 5 and plotted in figure 2.

Table 5. Surface Current. Minimum Values of $|b_i|$ for Stability, $m=0$

$\Lambda \backslash |b_e|$	0·0	0·5	1·0	1·5	2·0
5·0	0·707	0·700	0·677	0·637	0·577
4·0	0·707	0·695	0·658	0·592	0·483
3·0	0·707	0·685	0·612	0·468	0·000
2·0	0·707	0·645	0·408	0·000	0·000
1·5	0·707	0·548	0·000	0·000	0·000

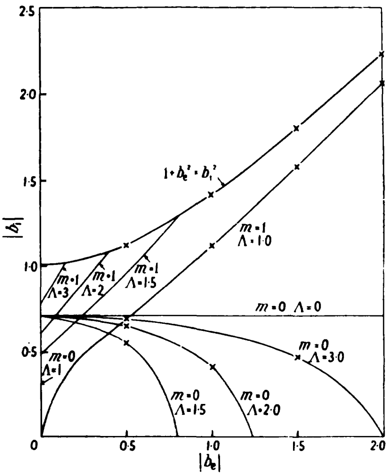

Figure 2. Stability diagram for surface current, with conducting walls. Stability occurs in the region above the curve marked with the given values of m and Λ and below the curve $1-b_e^2=b_i^2$. Complete stability for given Λ occurs when a point lies in the stability region for both $m=0$ and $m=1$.

Combining the results for $m=0$ and $m=1$, complete stability regions for four values of Λ (1·0, 1·5, 2·0, 3·0) are shown in figure 2. Thus as stated in § 1 complete stability can be obtained, but with two critical provisions: (*a*) wall radius is not too greatly different from discharge radius; (*b*) $|b_e|$ is very small unless Λ is very close to 1.

There is one other condition we may wish to apply. If the constricted discharge is to be a major element in the configuration the heat energy stored in the discharge should not be negligible compared with the total magnetic energy. Clearly it will be negligible if we go to a configuration in which Λ is approximately equal to unity and $|b_e|(\simeq|b_i|)\gg 1$; in this case the axial current and constriction are of relatively minor importance. In table 6 we tabulate the plasma energy ϵ_p as a fraction of the total energy for sets of ($|b_i|$, $|b_e|$, Λ) for which stability is obtained.

Table 6. Stable Configurations. Fraction of Energy Contained in Plasma

Λ	1·0	1·0	1·0	1·5	1·5	2·0	2·0	3·0	3·0	5·0
$|b_e|$	0·0	1·0	2·0	0·0	0·5	0·0	0·2	0·0	0·1	0·0
$|b_i|$	0·0	1·114	2·061	0·707	0·959	0·707	0·840	0·777	0·947	1·0
ϵ_p	1·000	0·478	0·210	0·364	0·195	0·285	0·185	0·175	0·051	0·000

§ 4. Configurations with Currents not Confined to the Surface

We consider now the case which arises if the initial rate of current rise of the discharge is not high enough for the surface currents and separated fields of the last section to be realized; similarly these conditions will be obtained when the initially separated fields have had time to interpenetrate. We obtain the equations for the case of a completely general distribution of axial currents and magnetic fields, but in this paper we consider only the case of an incompressible fluid. This restriction is made purely for algebraic simplicity since in the problems studied in I stability criteria were insensitive to this assumption. There is no difference in principle and the equations for the compressible fluid can be treated by the methods of this section.

We assume that in equilibrium we have an incompressible ideally conducting fluid of density ρ_0, forming a cylinder of radius r_0. The fluid carries a magnetic field $\mathbf{B}_0$ and current $\mathbf{j}_0$ and has pressure p_0, where $\mathbf{B}_0$, $\mathbf{j}_0$ and p_0 have the radial dependence given by

$$\mathbf{B}_0 = B_{\theta 0}\,[0, f(r)/f(r_0),\; bg(r)/g(r_0)], \qquad \ldots\ldots(4.1)$$

$$\mathbf{j}_0 = \frac{cB_{\theta 0}}{4\pi}\,[0, bg'/g_0, (rf)'/rf_0] \qquad \ldots\ldots(4.2)$$

and

$$p_0 = \frac{B_{\theta 0}{}^2}{8\pi}\,[b^2(1-g^2/g_0{}^2)+1-f^2/f_0{}^2+2\int_r^{r_0}(f^2/rf_0{}^2)\,dr] \qquad \ldots\ldots(4.3)$$

where the prime denotes differentiation with respect to r. In equations (4.2) and (4.3) and in what follows we write f for $f(r)$ and f_0 for $f(r_0)$ and similarly for the other functions. We have chosen the constant in equation (4.3) so that the pressure vanishes at the plasma surface, and the plasma is surrounded by a vacuum carrying a magnetic field $(0, B_{\theta 0}r_0/r, B_{\theta 0}b)$; as mentioned in I it is more satisfactory to assume that the outer region has a small but finite density, but the stability criteria are only slightly affected by such an assumption.

A perturbation of the form (2.1) is now applied, and from equations (2.6), (2.8) and (2.10) we find that the perturbed magnetic field can be expressed in terms of the perturbed velocities in the form

$$\mathbf{B}_1 = \frac{iB_{\theta 0}}{\omega r}\left[\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)v_{1r}, \left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)v_{1\theta}+\frac{ir^2}{f_0}\left(\frac{f}{r}\right)'v_{1r},\right.$$
$$\left.\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)v_{1z}+\frac{ibrg'}{g_0}v_{1r}\right]. \qquad \ldots\ldots(4.4)$$

The second and third components of equation (2.2) (using equations (4.1), (4.2), (4.4) and (2.4)) now give two equations for p_1 in terms of the components of v_1:

$$v_{1\theta}\left[Y_0{}^2+\frac{bkr_0{}^2g}{rg_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)\right]-v_{1z}\left[\frac{bmr_0{}^2g}{r^2g_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)\right]$$
$$+\frac{imY_0{}^2}{\rho_0\omega r}p_1-iv_{1r}\left[\frac{r_0{}^2(rf)'}{r^2f_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)+\frac{b^2mr_0{}^2gg'}{rg_0{}^2}-\frac{bkr_0{}^2rg}{f_0g_0}\left(\frac{f}{r}\right)'\right]=0, \qquad \ldots\ldots(4.5)$$

and

$$v_{1z}\left[Y_0{}^2+\frac{mr_0{}^2f}{r^2f_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)\right]-v_{1\theta}\left[\frac{kr_0{}^2f}{rf_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)\right]$$
$$+\frac{ikY_0{}^2}{\rho_0\omega}p_1-iv_{1r}\left[\frac{br_0{}^2g'}{rg_0}\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)+\frac{b^2mr_0{}^2fg'}{rf_0g_0}-\frac{kr_0{}^2rf}{f_0{}^2}\left(\frac{f}{r}\right)'\right]=0, \qquad \ldots\ldots(4.6)$$

where we have introduced the dimensionless variable $Y_0^2 = 4\pi\rho_0\omega^2 r_0^2/B_{\theta 0}^2$. We eliminate p_1 between equations (4.5) and (4.6) to obtain

$$[krv_{1\theta} - mv_{1z}]\left[Y_0^2 + \frac{r_0^2}{r^2}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)^2\right] - 2ikrv_{1r}\frac{r_0^2 f}{r^2 f_0}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right) = 0. \quad \ldots\ldots(4.7)$$

If this is combined with equation (2.10) we obtain an equation between v_{1r} and $v_{1\theta}$

$$m(rv_{1r})' + \frac{2k^2r^2\dfrac{r_0^2 f}{r^2 f_0}\left(\dfrac{mf}{f_0} + \dfrac{bkrg}{g_0}\right)}{\left[Y_0^2 + \dfrac{r_0^2}{r^2}\left(\dfrac{mf}{f_0} + \dfrac{bkrg}{g_0}\right)^2\right]} v_{1r} + iv_{1\theta}[m^2 + k^2r^2] = 0. \quad \ldots\ldots(4.8)$$

The first component of the equation of motion (2.2) yields a highly complicated equation between p_1, v_{1r}, $v_{1\theta}$, v_{1z} and their first derivatives. However p_1 and v_{1z} can be eliminated using equations (4.5) and (4.6) and we obtain a further equation between v_{1r} and $v_{1\theta}$:

$$\frac{2r_0^2 f}{r^2 f_0}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)(rv_{1r})' + m\left[Y_0^2 + \left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)\left(\frac{2r_0^2 f'}{mrf_0} - \frac{2r_0^2 f}{mr^2 f_0} + \frac{mr_0^2 f}{r^2 f_0} + \frac{bkr_0^2 g}{rg_0}\right)\right.$$
$$\left. + \frac{2bkr_0^2 fg'}{mf_0 g_0}\right] v_{1r} + i\left[Y_0^2 + \frac{r_0^2}{r^2}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)^2\right](rv_{1\theta})'$$
$$+ i\frac{2r_0^2}{r}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)\left(\frac{mf'}{f_0} + \frac{bkrg'}{g_0}\right)v_{1\theta} = 0. \quad \ldots\ldots(4.9)$$

We now make the substitutions

$$\left.\begin{aligned} &mv_{1r} = \phi \text{ and } m\beta v_{1r} + iv_{1\theta} = \chi \\ &\text{where} \quad \beta = \frac{2r_0^2 f}{r^2 f_0}\left[\frac{mf}{f_0} + \frac{bkrg}{g_0}\right] \Big/ m\left[Y_0^2 + \frac{r_0^2}{r^2}\left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)^2\right]. \end{aligned}\right\} \quad \ldots\ldots(4.10)$$

The variables ϕ and χ are not immediately relevant to the case $m = 0$ but that case can simply be treated separately. Using (4.10), equations (4.8) and (4.9) become

$$(r\phi)' + (m^2 + k^2r^2)\chi - m^2\beta\phi = 0 \quad \ldots\ldots(4.11)$$

$$(r\chi)' + m\beta\left[\frac{rf_0}{f}\left(\frac{mf'}{f_0} + \frac{bkrg'}{g_0}\right)\right]\chi$$
$$+ \left[1 - \left\{\frac{m(rf' - f)}{f_0} \Big/ \left(\frac{mf}{f_0} + \frac{bkrg}{g_0}\right)\right\}\beta - m^2\beta^2\right]\phi = 0. \quad \ldots\ldots(4.12)$$

We can now apply the boundary conditions (2.14), (2.15) at the perturbed plasma surface and obtain the dispersion relation between m, k and ω in terms of the values ϕ_0, χ_0 of ϕ and χ at $r = r_0$. Thus

$$\frac{\chi_0}{\phi_0} = -\frac{(m + bkr_0)^2}{kr_0[Y_0^2 + (m + bkr_0)^2]}\frac{[K_m(kr_0)I_m'(kR_0) - I_m(kr_0)K_m'(kR_0)]}{[K_m'(kr_0)I_m'(kR_0) - I_m'(kr_0)K_m'(kR_0)]}. \quad \ldots\ldots(4.13)$$

We again introduce dimensionless variables (3.11) to obtain

$$\frac{\chi_0}{\phi_0} = -\frac{(m + bX_0)^2}{X_0[Y_0^2 + (m + bX_0)^2]}\frac{[K_m(X_0)I_m'(\Lambda X_0) - I_m(X_0)K_m'(\Lambda X_0)]}{[K_m'(X_0)I_m'(\Lambda X_0) - I_m'(X_0)K_m'(\Lambda X_0)]}. \quad \ldots\ldots(4.14)$$

Although the equations (4.11) and (4.12) and the dispersion relation (4.14) are relatively simple, it is clear that any general solution of them must be obtained numerically. However we can make some general comments. We are interested in the possible existence of functions f and g and constants b and Λ so that for no values of X_0 and m can the dispersion relation have solutions for positive values of Y_0^2. Such functions f and g must yield values of p_0 which are consistent with the idea of a constricted gas discharge and give field configurations which might be expected to arise naturally.

If $Y_0^2>0$, the right-hand side of equation (4.14) is positive; thus ϕ_0 and χ_0 must have the same sign. In general ϕ and χ have opposite signs at $r=0$. (The only non-singular exception is when there is a uniform distribution of axial current near the discharge centre.) ϕ cannot change sign before χ and χ cannot change sign while

$$1-\left\{\frac{m(rf'-f)}{f_0}\bigg/\left(\frac{mf}{f_0}+\frac{bkrg}{g_0}\right)\right\}\beta-m^2\beta^2\geqslant 0. \qquad \ldots\ldots(4.15)$$

This condition can be reduced to the form

$$\left(m+\frac{bkrgf_0}{fg_0}\right)^2\geqslant\frac{2(rf'+f)}{f}=\frac{2r}{I_z}\frac{dI_z}{dr} \qquad \ldots\ldots(4.16)$$

where $I_z(r)$ is the total axial current contained within radius r. Equation (4.16) is a generalization of one that was given in I. Thus in the absence of an axial field instability cannot occur if

$$m^2\geqslant\left(\frac{2r}{I_z}\frac{dI_z}{dr}\right)_{\max} \qquad \ldots\ldots(4.17)$$

where the suffix means that we must take the maximum value of the quantity between $r=0$ and $r=r_0$. Note that this sufficiency condition can give no information about modes with $m<2$ as for non-singular current distributions

$$\left(\frac{2r}{I_z}\frac{dI_z}{dr}\right)_{\max}\geqslant 4.$$

It is clear that the inequality cannot be as useful when the axial field is included; for, if b is negative and if f and g are given, there must be combinations of m and k for which it fails. There are however some values of m and k for which there must be stability, and for other values of m and k the inequality may only fail to be satisfied for a narrow range in r; in this case χ must change sign within this range if instability can occur. The inequality itself is not a very restrictive condition and it can be shown that in interesting cases χ is still increasing at the time (4.16) fails to be satisfied; thus it must have a turning value before it can vanish. These general results should guide the numerical work which we hope will follow the present paper.

§ 5. Uniform Axial Field and Current

Now we turn our attention to one special case for which a full solution can be obtained. If the axial field and current are both uniform ($g\propto 1$, $f\propto r$) equation (4.12) reduces to the simpler form

$$(r\chi)'+m^2\beta\chi+(1-m^2\beta^2)\phi=0 \qquad \ldots\ldots(5.1)$$

where β is now a constant. We generalize the problem slightly to allow for differing internal and external axial fields, b_i, b_e ($b_\mathrm{i}^2<b_\mathrm{e}^2$). Thus

$$\beta=2(m+b_\mathrm{i}kr_0)/m\,[Y_0^2+(m+b_\mathrm{i}kr_0)^2]$$

3 Y-2

and the dispersion relation becomes

$$\frac{\chi_0}{\phi_0} = -\frac{(m+b_e X_0)^2}{X_0[Y_0^2+(m+b_i X_0)^2]} \frac{[K_m(X_0)I_m'(\Lambda X_0) - I_m(X_0)K_m'(\Lambda X_0)]}{[K_m'(X_0)I_m'(\Lambda X_0) - I_m'(X_0)K_m'(\Lambda X_0)]}. \qquad \ldots\ldots(5.2)$$

Table 7. Volume Current. Equal Internal and External Fields, no Walls. Table of Wave Numbers and Corresponding Growth Rates.

X_0 \ b	0·0	0·5	−0·5	1·0	−1·0	2·0	−2·0	5·0	−5·0
0·05								−0·6072	0·3796
0·10							0·3403	−1·3851	0·5062
0·15								−2·2175	0·3779
0·20				−0·3649		−0·9531	0·5059		0·0000
0·25		−0·1450			0·4315				−0·0302
0·30						−1·5089	0·4997		−0·1730
0·35									−0·4290
0·40				−0·7054		−2·0876	0·3269		
0·50	0·2481	−0·2169	0·5088		0·5575		0·0000		
0·60				−1·0455			0·0212		
0·70							−0·0193		
0·75		−0·2826			0·3959				
0·80				−1·4083					
1·00	0·4670	−0·3645	0·6090	−1·8111	0·0000				
1·20					0·0778				
1·40					0·1067				
1·50		−0·6054	0·4095						
1·60					0·0834				
1·80					0·0048				
2·00	0·6935	−0·9650	0·0000						
2·50			0·2053						
3·00			0·3447						
3·50	0·8319		0·4023						

In obtaining these results care is required to find the largest root of the equation (5.2) for Y_0^2 (Tayler 1957 b).

Except that the definition of β has been altered the equations (4.11) and (5.1) are exactly those found in I in the absence of an axial field and their solution is

$$\phi \propto \frac{m^2\beta J_m[(m^2\beta^2-1)^{1/2}kr]}{(m^2\beta^2-1)kr} + \frac{J_m'[(m^2\beta^2-1)^{1/2}kr]}{(m^2\beta^2-1)^{1/2}},$$

$$\chi \propto \frac{J_m[(m^2\beta^2-1)^{1/2}kr]}{kr}..$$

We have considered full solutions for $m=1$ which was the most difficult mode to stabilize for surface currents. It is immediately fairly easily seen that there can be no complete stability. At the point $1+b_e X_0=0$ the largest root for Y_0^2 corresponds to $\beta=1$. Thus $Y_0^2=1-b_i^2/b_e^2$, and this is positive unless $b_i=b_e$. If $b_i=b_e$ a perturbation expansion shows that Y_0^2 is decreasing with X_0 at this point and is positive for smaller X_0. Thus stability cannot be expected for completely penetrated fields. Qualitatively the results are only slightly affected by the presence of conducting walls although some stability is produced at long wavelengths. Results for $m=1$ are shown in tables 7 and 8 and figure 3. In table7 and figure 3 are given results for $\Lambda=\infty$ (no walls) and several values of $b_i(=b_e)$. In table 8 are shown results for $b_e=-1$ and several values of b_i and Λ. The results may be expected to be qualitatively true for a compressible fluid although the discontinuity in the growth rate against wave number curve at $b_i X_0 = -1$ should be removed.

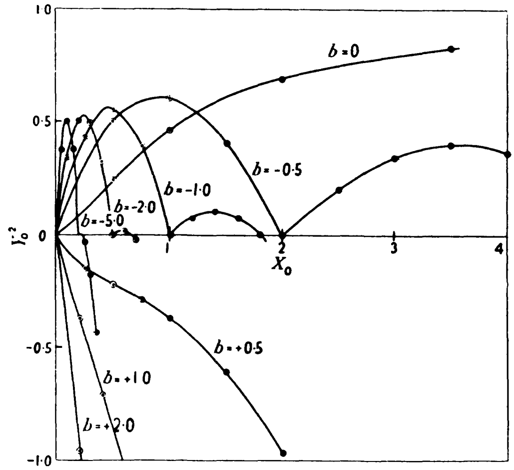

Figure 3. Volume current. Plot of growth rate against wave number for the case of equal internal and external fields. Curves are plotted for eight values of $b_e (= b_i)$; they show the square of the dimensionless growth rate $(Y_0^2 = 4\pi\rho_0\omega^2 r_0^2/B_{\theta 0}^2)$ as a function of the dimensionless wave number $(X_0 = kr_0)$.

Table 8. Volume Current. Internal and External Fields and Conducting Walls. Table of Wave Numbers and Corresponding Growth Rates; $b_e = -1$

b_i	Λ	X_0: 0·0	0·2	0·4	0·6	0·8	1·0	1·2	1·4	1·6	1·8	2·0
−1	∞	0·0000	0·3673	0·5449	0·5240	0·3341	0·0000	0·0778	0·1067	0·0834	0·0048	< 0
−1	2	−0·6667	−0·0204	0·3795	0·4741	0·3260	0·0000	0·0775	0·1061	0·0824		< 0
0	∞	0·0000	0·4063	0·6970	0·8829	0·9738	1·0000	0·9788	0·9240	0·8485	0·7667	0·6935
0	2	−0·6667	0·0134	0·5334	0·8314	0·9656	1·0000	0·9756	0·9164	0·8387	0·7573	0·6861

b_i	Λ	$1/X_0$: 1·0	0·9	0·8	0·7	0·6	0·5	0·4	0·3	0·2	0·1	0·0
0	∞	1·0000	0·9931	0·9677	0·9140	0·8212	0·6935	0·6054	0·6575	0·8044	0·9443	1·0000
0	2	1·0000	0·9919	0·9634	0·9060	0·8112	0·6861	0·6034	0·6574	0·8044	0·9443	1·0000

§ 6. Conclusions and Further Problems

We can summarize the mathematical results of the last two sections by stating that stable configurations may exist with the following properties:

(*a*) The conducting walls must be relatively close to the plasma and most favourable results are obtained if the axial magnetic field is mainly trapped within the plasma.

(*b*) There is thus no stability against small disturbances for highly constricted discharges of the type considered by Pease (1957); we have not however ruled out the possibility that such instabilities could be limited at finite amplitude.

(c) The most stable configurations in the absence of external fields are not the most easily stabilized in their presence; thus a uniform current distribution was the most stable configuration discussed in I but the fields considered in the present paper cannot make it completely stable.

Before we can state that physically stable configurations can exist it is of importance to obtain results for the case of partially interpenetrated fields, and to be able to answer the following two questions :

(a) How closely can the axial and azimuthal fields be separated in an experimental arrangement?

(b) How long can a stable configuration be held against dissipative processes? In principle any answer to the second question must be related to a study of the rate of penetration of fields into a highly ionized plasma, though an order of magnitude estimate can be obtained by taking a rigid conductor of similar conductivity. It is hoped to obtain some results using a digital computer. There are still obviously many approximations in this theory. We can conveniently divide them into two classes, (a) geometrical and (b) physical.

Geometrical approximations.

(a) These arise because actual experiments are not performed in infinite cylinders. The photographs of Carruthers and Davenport (1957) show discharges in a cylindrical tube with electrodes and in a toroidal tube. In addition metal walls may not be continuous but may contain slits for the introduction of fields.

No direct attempt is made in this paper to discuss the possibility of new instabilities introduced by toroidal geometry and the curvature of the equilibrium current channel. An attempt can be made to approach finite geometry by applying periodic boundary conditions to results already obtained in an infinite cylinder; this places an upper limit on the possible instability wavelengths. Thus we can formally obtain a completely stable configuration for the case of surface currents, provided that the critical wavelength below which there is stability (shown in table 2) is greater than the length of the system L. For X_0 to be small enough for the resultant dimension to be practicable we must be in the part of the stability diagram (figure 1) where $|b_e| \gg 1$, $|b_i| \simeq |b_e|$. Under these conditions there is complete stability provided that $b_i > L/2\pi r_0$.

This is however a situation in which the external field considerably exceeds the discharge field and not one in which we are strictly interested in this paper.

The problem of periodic slits in an infinite conducting wall can be treated by the method of this paper but the resulting dispersion relation is in the form of a very complicated infinite determinant. However, qualitatively it appears that narrow slits should not have a serious effect on an otherwise completely stabilized discharge. Even in the absence of walls only long wavelength instabilities remain; the narrow slits alter these normal modes by the introduction of higher harmonics of wavelength comparable with the slit width and these are prevented from growing by the trapped axial field.

Physical approximations.

(b) Obvious physical approximations are the neglect of all dissipative terms and especially the artificial division of the fluid into regions of infinite and zero electrical conductivity. One such neglected effect which might lead to slightly enhanced stability is the anisotropic electrical conductivity of the discharge; this causes an axial magnetic field to increase towards the discharge centre.

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Stability of a Linear Pinch

By Bergen R. Suydam *

It is now well known that a trapped longitudinal magnetic field has a stabilizing influence on a linear pinch. Once such a stabilized pinch is set up, however, diffusion will lead to mixing of the initially crossed fields; the torsion of the field lines will diminish and the plasma may ultimately become unstable. It is the purpose of this paper to study continuous plasma and field distributions in order to see at what point instability might be expected.

A variational principle has been given [1] which applies very nicely to the problem at hand. Briefly, one subjects the plasma to a displacement $\boldsymbol{\xi}$ and calculates the resulting change in the total energy, δW, of the hydromagnetic system. Stability then hinges on whether or not some $\boldsymbol{\xi}$ can produce a diminution of the energy. If we define the vector $\mathbf{Q} = \boldsymbol{\nabla}\times[\boldsymbol{\xi}\times\mathbf{B}]$ it turns out that the change in energy is

$$\delta W = \int d^3x\{\mathbf{Q}\cdot\mathbf{Q} - 4\pi\mathbf{J}\cdot[\mathbf{Q}\times\boldsymbol{\xi}] + \gamma p(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p) \quad (1)$$

where γ is the specific heat ratio and p is the pressure. The integration is taken over the complete volume. If a displacement exists which makes δW negative, we have instability. In the linear pinch we are dealing with axial symmetry and the components of the magnetic field in cylindrical coordinates are $\mathbf{B} = (0, B_\theta, B_z)$. Moreover, it is assumed that B_θ, B_z and p are functions of r alone. It is then possible to analyze $\boldsymbol{\xi}$ in terms of displacements of the form

$$\boldsymbol{\xi} = [\xi_r(r), \xi_\theta(r), \xi_z(r)]\exp i(kz + m\theta). \quad (2)$$

The integration with respect to θ and z can be carried out and δW can be minimized with respect to ξ_θ and ξ_z by purely algebraic means. When this computation has been carried out we find

$$\delta W = \int_{r_1}^{r_2}\left\{\frac{(rf\xi' + g\xi)}{m^2 + (kr)^2} + (f^2 - h)\xi^2\right\}2rdr, \quad (3)$$

where we have set

$$f \equiv kB_z + mB_\theta/r$$
$$g \equiv kB_z - mB_\theta/r$$
$$h \equiv (8\pi J_z/r)B_\theta$$
$$\xi \equiv d\xi'/dr,$$

and have dropped the subscript r on ξ for simplicity.

In order that δW be a minimum with respect to functions ξ, the displacement must be a solution of the Euler-Lagrange equation

$$\xi'' + P\xi' + Q\xi = 0,$$

where

$$P \equiv 3/r + 2f'/f - 2k^2r/[m^2 + (kr)^2] \quad (6)$$
$$Q \equiv -[(kr)^2 + (m^2 - 1)]/r^2 - 2k^2g/f[m^2 + (kr)^2] - 8\pi k^2p'/rf^2.$$

When ξ is chosen to be a solution of Eq. (5), the integrand of Eq. (3) is a perfect differential and we have

$$\delta W = 2\left[\frac{r^2f\xi(rf\xi' + g\xi)}{m^2 + (kr)^2}\right]_{r_1}^{r_2}. \quad (7)$$

Equation (5) must be solved subject to certain boundary conditions which we take to be

$$\xi(0) \text{ is finite,}$$
$$\xi(R) = 0. \quad (8)$$

In choosing the second boundary condition we have placed a perfectly conducting wall at $r = R$ in order to benefit from its stabilizing influence.[2, 3, 4] However, it will turn out that the wall has no stabilizing effect on the modes we shall study.

Our previous experience with the theory of instabilities [5] leads us to be particularly wary of "fluted" displacements which interchange magnetic field lines without bending them. The bending of field lines requires energy while interchanging them does not. The purpose of the trapped axial field was to stabilize the plasma by twisting the field lines so that any arbitrary displacement will bend some of them. Nevertheless, some displacements will bend some field lines less than others and it seems reasonable to expect that those displacements which bend the lines the least will be the most dangerous.

Now the magnetic field lines describe a set of spirals with a pitch

$$\mu \equiv B_\theta/rB_z \quad (9)$$

which, in general, varies from layer to layer. The lines of the $\boldsymbol{\xi}$ field (2), on the other hand, describe a set of spirals whose pitch is constant. If these two sets of spirals should match over a finite region of

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space then a displacement is possible which does not bend magnetic field lines. If, on the other hand, μ' is not zero anywhere, then it is nevertheless always possible to choose k such that the two spiral systems match at a particular radius. When this happens, displacements are possible which bend magnetic field lines very little in the neighborhood of this point. Accordingly, we shall assume that the worst choice of k, m is such that

$$f = kB_z + mB_\theta/r \tag{10}$$

vanishes at some point in (0, R). Let the point where $f = 0$ be denoted by $r = a$.

Now we note that $r = a$ is a regular singular point of Eq. (5). The theory of such singularities tells us that the solutions to Eq. (5) can be written in the form

$$\xi = (r - a)^\nu \times \text{Power series in } (r - a),$$

where ν is a root of the indicial equation

$$\nu^2 + \nu + M^2 = 0, \tag{11}$$

with

$$M^2 \equiv \frac{8\pi p'}{rB_z^2}\left(\frac{\mu}{\mu'}\right)^2\Bigg|_{r=a}.$$

Thus we have

$$\nu_{1,2} = \tfrac{1}{2}(1 \pm (1 - 4M^2)^{1/2}). \tag{12}$$

If the roots are real we have $4M^2 \leqslant 1$. Now the boundary conditions (8) determine ξ uniquely (except for a normalization factor) in $(0 - a)$ and in $(a - r)$. Therefore, on either side of $r = a$ there will be an admixture of the more singular of the two solutions; i.e.

$$\xi = (r - a)^{\nu_1}\{1 + \text{higher order terms in } (r - a)\}, \tag{13}$$

where ν_1 is given by Eq. (12) with the minus sign. For this choice of ξ the integral (3) diverges and ξ is an improper function.

We can, however, consider the following proper displacement: ξ is the (properly normalized) solution of Eq. (5) in the ranges $0 \leqslant r \leqslant a - \epsilon$ and $a + \epsilon \leqslant r \leqslant R$. ξ is constant for $a - \epsilon < r < a + \epsilon$. If ϵ is chosen so small that the first term of Eq. (13) dominates the power series, then this first term can be substituted into Eq. (1) and it follows that δW is always positive.

If the roots are complex, it is convenient to write Eq. (12) in the form

$$\nu_{1,2} = -\tfrac{1}{2}(1 \pm i\beta)$$
$$\beta \equiv (4M^2 - 1)^{1/2}.$$

In terms of β, the displacement ξ is given by

$$\xi = |r - a|^{-1/2}\cos\{\tfrac{1}{2}\beta\log|r - a| + \phi\} \times [1 + \text{higher order terms}], \tag{15}$$

where ϕ is a constant phase angle determined by the boundary conditions. Again $\xi(r)$ is an improper function at $r = a$. We can circumvent this difficulty by choosing a displacement given by Eq. (15) over the range $0 \leqslant r \leqslant a - \epsilon$ and setting $\xi =$ constant for $a - \epsilon \leqslant r \leqslant a$. The range $a \leqslant r \leqslant R$ is treated in a similar fashion. The quantity ϵ is to be chosen so that the leading term in Eq. (15) dominates at $r = a - \epsilon$. When this choice of ξ is made we obtain

$$\int_0^a \left\{\frac{(rf\xi' + g\xi)^2}{m^2 + (kr)^2} + (f^2 - h)\xi\right\} 2rdr = \frac{a^3 B_z^2 \mu'^2}{2(1 + (a\mu)^2)}\Big[1 - 2M^2 + (1 - 2M^2)\cos 2\psi + \beta\sin 2\psi\Big], \tag{16}$$

where ψ is defined by

$$\psi \equiv \tfrac{1}{2}\beta\log\epsilon + \phi \tag{17}$$

and can be made anything we please (modulo 2π) by a suitable choice of ϵ. But the bracketed expression on the righthand side of Eq. (16) oscillates between the values 1 and $(1 - 4M^2)$ as ψ varies. If $(1 - 4M^2)$ is negative, it is possible to choose ϵ so that (16) is negative. Similarly, the integral taken from a to R can be made negative and we have found that complex roots imply instability.

The result of our investigation can be stated as a theorem:

A necessary condition that the $m \neq 0$ modes of a linear pinch be stable is that

$$(r/4)(\mu'/\mu)^2 + 8\pi p'/B_z^2 \geqslant 0 \tag{18}$$

at every point in the plasma.

The method by which we obtained this theorem from the Euler-Lagrange equation suggests the likelihood that the above inequality might also be a sufficient condition for stability. However, there are two major difficulties which will be discussed in the following paragraphs.

The foregoing analysis, leading to our theorem, suggests the importance of extremely localized mixing at any point of instability since the unstable modes we have found are those for which the radial displacement ξ is very small except in the immediate neighborhood of $r = a$, where the B and ξ fields interlace. Therefore, it is of interest to inquire: suppose a small region is unstable, in the sense that the inequality stated in our theorem is violated, what then happens?

The answer to such a question is very difficult to give, but an estimate has been made in the following manner: A displacement $\boldsymbol{\xi}$ of the unstable type is chosen. This leads to new values for p, B_z, B_θ and ρ in the neighborhood of $r = a$. Mixing is now simulated by replacing p, ρ, B_θ, B_z by the values obtained by averaging over θ, and we ask whether the new distribution is more or less stable using the above theorem as a criteria.

The result seems to go qualitatively as follows: The mixing of a small unstable region leads to a distribution which is less unstable on the inside and more unstable on the outside. Thus, if some interior shell were unstable, it would mix until stable and this, in turn, would upset the stability of the next shell which would proceed to mix and so on. In this fashion such an instability would propagate towards the surface. If, however, a layer near the surface is given excess stability, the outward progression of the mixing should be stopped. This excess stability of the surface layers ought to be insured if the B_z field were so programmed that μ' is made quite large in this region. The simplest programming appears to be one which would reverse B_z in the vacuum after the plasma has pinched.

The inequality appearing in our theorem is a necessary condition for stability. It might also be argued that this is a sufficient condition since it was obtained from the Euler-Lagrange equation of (3). There are, however, two difficulties to be overcome before this can be asserted. The first is that we did not truly minimize δW with respect to k, but rather used a heuristic *principle of minimum bending*. However, the principle seems physically reasonable and the objection does not seem to be very serious. The second difficulty is quite deep and arises because the Euler-Lagrange equation is a necessary condition, but in no way guarantees a minimum.

Some progress has been made in clarifying the second difficulty. Note that the condition that $\xi(0)$ be finite determines a solution of the Euler-Lagrange equation, and that $\xi(R) = 0$ determines another solution. It has been possible to prove the following theorem:

The necessary and sufficient condition for stability is that neither of the above-mentioned solutions to the Euler-Lagrange equation has a zero in the open interval $(0, R)$.

This makes further progress possible by application of the Sturmian theory to Eq. (5).

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Hydromagnetic Stability of a Diffuse Linear Pinch*

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The hydromagnetic energy principle is applied to the derivation of necessary and sufficient conditions for the hydromagnetic stability of a linear pinch with distributed plasma current (a diffuse linear pinch). The results are quite general in that the axial and azimuthal components of the magnetic field, which determine the structure of the pinch completely, are treated as arbitrary functions of distance from the axis. For purposes of illustration, the general results are applied to the limiting case of a pinch with the plasma current confined to an infinitely thin layer (a sharp pinch).

I. INTRODUCTION

We shall investigate the hydromagnetic stability of a diffuse linear pinch (diffuse in the sense of a spread-out plasma current distribution; the sharp pinch, in which the plasma current is confined to an infinitely thin layer, is included as a limiting case). For this purpose we take as an idealized model any cylindrically symmetrical and infinitely long configuration of plasma and magnetic field with the following properties: (1) The plasma has infinite conductivity. (2) The magnetic field has axial and azimuthal components, but no radial component. (3) The plasma stress tensor is isotropic[1]; we may therefore speak of a scalar plasma pressure P. (4) The plasma pressure gradient is balanced by the magnetic force $\mathbf{J} \times \mathbf{B}$, where $\mathbf{J} = \nabla \times \mathbf{B}$ is the plasma current density. (5) The system is bounded on the outside by a perfectly conducting wall of radius b. (6) It may also be bounded on the inside by a perfectly conducting wire of radius a. If so, it is called a tubular pinch; if not, a columnar pinch. In either case, if r = distance from the axis, the pinch occupies the space $a < r < b$, where $a = 0$ for a columnar pinch. (7) There are no vacuum regions.[2] There may, however, be regions in which the plasma pressure is negligibly small even though the conductivity is still infinite.

Let us introduce cylindrical coordinates r, θ, z about the axis of symmetry. The field component B_r vanishes, and the components B_θ and B_z depend only on

* This work was performed under auspices of the U. S. Atomic Energy Commission.

[1] The isotropy holds only for the equilibrium state; it is not preserved by small oscillations.

[2] This restriction will be removed later on. (See Theorem 14 below.)

r. (The field lines, consequently, form a system of coaxial helices with pitch $2\pi r B_z/B_\theta$.) The current density in the plasma is determined by Maxwell's equation $\mathbf{J} = \nabla \times \mathbf{B}$; the radial component J_r therefore vanishes, and the other components are given by

$$J_\theta = -\frac{dB_z}{dr}; \qquad J_z = \frac{1}{r}\frac{d}{dr}(rB_\theta). \tag{1}$$

The radial distribution of the plasma pressure P is governed by the magnetohydrostatic equilibrium condition $\nabla P = \mathbf{J} \times \mathbf{B}$, which reduces to

$$\frac{dP}{dr} + B_z\frac{dB_z}{dr} + \frac{B_\theta}{r}\frac{d}{dr}(rB_\theta) = 0. \tag{2}$$

If B_θ and B_z are given as functions of r, then J_θ, J_z, and P, the other quantities characterizing the equilibrium state, are determined by Eqs. (1) and (2), except for an additive constant in P. Since we are not assuming any particular boundary condition to be satisfied by P at the walls, the additive constant will remain undetermined, but it will be clear later on that the stability of a pinch does not depend on this constant. For our purposes, therefore, the structure of a pinch can be specified completely by giving the functions $B_\theta(r)$ and $B_z(r)$ between the limits a and b. We shall not assume any special form for these functions. Instead, we shall allow them to be completely arbitrary in order to obtain results general enough to be applied to any diffuse linear pinch.[3]

So far we have considered only the equilibrium state of a pinch, but in order to derive stability criteria we must also examine the behavior of small displacements from this equilibrium state. According to hydromagnetic theory, any such displacement can be described by a single vector function of position, the displacement $\boldsymbol{\xi}$ of the plasma itself (*1*). The displacement in the magnetic field, for example, is determined by the fact that the field lines are constrained to move with the plasma (*2*, *3*).

Our treatment will be based on the hydromagnetic energy principle, which was first derived in its most general form by Bernstein *et al.* (*1*).[4] According to this principle, a system is stable if a certain energy integral $W(\boldsymbol{\xi})$ is positive for every displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, unstable if there exists a $\boldsymbol{\xi}$ for which $W(\boldsymbol{\xi})$ is negative. The energy integral is given by

$$W(\boldsymbol{\xi}) = \tfrac{1}{2}\int d^3x\, \{Q^2 + \mathbf{J}\cdot\boldsymbol{\xi}\times\mathbf{Q} + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}\cdot\nabla P + \gamma P(\nabla\cdot\boldsymbol{\xi})^2\}, \tag{3}$$

[3] As is now evident, we are using the term "diffuse pinch" in such a way as to include a variety of configurations that would not ordinarily be described as pinches, e.g., stellarator-type configurations in which the plasma is contained by a purely axial field.

[4] See also Lundquist (*4*) for an earlier and less general form of the energy principle.

where $\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ and γ is the ratio of specific heats. Ordinarily, the most straightforward way of applying the energy principle is to impose a conveniently chosen normalization condition on $\boldsymbol{\xi}$ and then to minimize $W(\boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$; the system is then stable if the minimum value is positive and unstable if it is negative.

The energy principle says nothing about the marginal case where a nontrivial $\boldsymbol{\xi}$ exists for which $W(\boldsymbol{\xi}) = 0$, but none for which $W(\boldsymbol{\xi}) < 0$. When this happens the system can be either stable, unstable, or neutrally stable, as we can see from the simple example of a particle moving in the one-dimensional potential $V(x) = \alpha x^2 + \beta x^4$. The term αx^2 corresponds to $W(\boldsymbol{\xi})$, and the equilibrium point at $x = 0$ is stable or unstable according as α is positive or negative. If $\alpha = 0$, the equilibrium is stable, unstable, or neutrally stable according as β is positive, negative, or zero. Fortunately, we shall soon see that the question of what happens in the marginal case is of no physical interest for a linear pinch.

We note, by way of justification, that the hydromagnetic energy principle is, in the case of a diffuse linear pinch with an isotropic plasma stress-tensor,[5] identical with the energy principle derived from the Boltzmann equation in the limit of small m/e (*6–9*).[6] Nevertheless, it does not take into account the effects of runaway electrons (*10*), negative Landau damping (*11*), or nonadiabatic particle motions (*12*), all of which may give rise to instabilities that are not predicted by our analysis.

It follows from symmetry considerations that the small-amplitude motions of a linear pinch can be analyzed into normal modes for which ξ_r, $i\xi_\theta$, and $i\xi_z$ are real functions of r multiplied by $\exp(im\theta + ikz)$. We can therefore limit ourselves to displacements of this form without loss of generality. Furthermore, the original derivation of the energy principle (*1*), which refers to the class of all possible displacements, remains valid even if the class of displacements is restricted by fixing m and k. A separate stability criterion is thus obtained for each set of values of m and k.

It is very easy to minimize the energy integral $W(\boldsymbol{\xi})$ with respect to the components ξ_θ and ξ_z. The result is a reduction of W to the one-dimensional form (*13*)

$$W(\xi) = \frac{\pi}{2} \int_a^b r\, dr\, \Lambda \left(\xi, \frac{d\xi}{dr}\right), \tag{4}$$

[5] The diffuse pinch with a nonisotropic stress tensor will be treated by Baños and Schwartz (*5*).

[6] Since the Boltzmann equation accounts for the deviations from isotropy associated with small oscillations, the agreement between the two energy principles implies that the stability criterion is unaffected by these deviations. This is true because the deviations vanish for the marginally stable normal modes, which form the boundary between stable and unstable regions. (An exceptional case to which these statements do not apply will be discussed in the next section.)

where ξ is an abbreviation for the radial component ξ_r and Λ is a certain quadratic form in ξ and $d\xi/dr$ with m and k as parameters. The pinch is then stable for given values of m and k if $W(\xi)$ is positive for every ξ and unstable if there exists a ξ for which $W(\xi)$ is negative. For a certain discrete set of values of m and k there will exist a nontrivial ξ for which $W(\xi)$ vanishes, but none for which $W(\xi)$ is negative. We do not know whether the pinch is stable for these marginal values of m and k, but it does not really matter because the pinch will definitely be stable for neighboring values of k on one side of the marginal value and unstable for neighboring values on the other side. For all practical purposes, therefore, the stability condition resulting from Eq. (4) may be regarded as necessary and sufficient.

Several years ago Rosenbluth (*14*) investigated the hydromagnetic stability of a sharp columnar pinch[7] with respect to displacements that are continuous across the current layer. He later found, however, that the restriction to continuous displacements is not correct, and by dropping this restriction he was able to derive a new and more stringent stability criterion (*21*). From the standpoint of the energy principle the difficulty amounts to this (*22*): some of the displacements for which the energy integral is negative in a diffuse pinch become discontinuous in the limit of an infinitely thin current layer. This is why the new stability criterion is more stringent than the old one—the class of allowed displacements is wider, since it now contains discontinuous as well as continuous displacements. The most striking feature of the new stability criterion is that it depends on the detailed structure of the current layer; one cannot simply treat J_θ and J_z as delta functions. For this reason it can only be derived by treating the sharp pinch as a limiting case of the diffuse pinch. This circumstance alone lends great interest to the theory of diffuse-pinch stability.

Suydam (*13*) has applied the hydromagnetic energy principle to a generalized diffuse pinch.[8] After reducing the energy integral to the one-dimensional form (4), he obtained the following necessary condition for stability:

$$\frac{r}{8} B_z^2 \left(\frac{d \log \mu}{dr}\right)^2 + \frac{dP}{dr} > 0, \tag{5}$$

where $\mu(r) = B_\theta/rB_z$. A pinch is stable for all values of m and k only if this inequality is satisfied for every value of r between a and b. (The quantity μ is simply 2π divided by the pitch of a field line.)

Suydam's necessary condition for stability is obviously not sufficient. Consider, for example, a sharp pinch that is unstable according to Rosenbluth's criterion. If the current layer is of infinitesimal thickness δ, then the first term in (5),

[7] Other treatments of the sharp columnar pinch are reported in Refs. *15–20*.

[8] For further information on the stability of a generalized diffuse pinch, see Refs. *17* and *18*. Stability criteria have also been derived for diffuse pinches with special current distributions, e.g., uniform J_z and vanishing J_θ (*17*, *18*, *23*).

which is necessarily positive, is of order δ^{-2} while the second term is only of order δ^{-1}. The pinch therefore satisfies Suydam's condition in spite of its instability.[9]

What is needed is a necessary and sufficient condition for stability in a form suitable for numerical calculation, and our main purpose will be to derive one by minimizing $W(\xi)$. We shall start in Section II by minimizing with respect to the components ξ_θ and ξ_z, thus reducing W to the one-dimensional form expressed by Eq. (4). The following comparison theorem will be an immediate consequence: A linear pinch is stable for all values of m and k if and only if it is stable for $m = 1$, $-\infty < k < \infty$ and for $m = 0$ in the limit as $k \rightarrow 0$. Another immediate consequence will be that the stability criterion is independent of γ, the ratio of specific heats. The next step, minimization with respect to ξ_r, will involve two difficulties. First, the Euler–Lagrange equation for this minimization problem is singular at certain values of r between a and b, and second, there are cases of interest in which the relevant Euler–Lagrange solution gives not the minimum but only a stationary value for W. The first difficulty will be surmounted by proving that W can be minimized separately in each of the intervals between singular points, and the second by proving that the pinch is unstable whenever the Euler–Lagrange solutions fail to give a minimum. The minimization will be completed in Section IV after a discussion in Section III of some elementary properties of the Euler–Lagrange solutions. Then, in Section V, the necessary and sufficient condition for stability[10] will be given in several forms, all of them depending on the behavior of the Euler–Lagrange solutions. Since the Euler–Lagrange equation can always be solved numerically, we will have found a definite procedure for determining the stability of any pinch configuration that has been specified by giving B_θ and B_z as functions of r. Finally, for illustrative purposes, we shall treat the sharp pinch in Section VI as a limiting case, obtaining stability criteria in closed form. The results of that section will agree with those of Rosenbluth for the sharp columnar pinch (*21*) and with those of Newcomb and Kaufman for the sharp tubular pinch (*22*).

II. REDUCTION OF THE ENERGY INTEGRAL TO A ONE-DIMENSIONAL FORM

As we have seen in the introduction, it is permissible to restrict our attention to displacements for which ξ_r, $i\xi_\theta$, and $i\xi_z$ are real functions of r multiplied by $\exp(im\theta + ikz)$, thus obtaining a separate stability criterion for each set of values of m and k. Assuming for the time being that m and k do not both vanish,

[9] Exceptional cases arising when $d\mu/dr$ vanishes at some point in the current layer are considered briefly in Section VI. (Points where B_z vanishes give no trouble because $d \log \mu/dr$ becomes infinite.)

[10] This condition will be a slight generalization of the one given by Rosenbluth (*21*) on the basis of a heuristic argument involving marginal stability.

and dropping the exponential factor, let us transform to the real variables ξ, η, ζ:

$$\xi = \xi_r, \tag{6a}$$

$$\eta = \nabla \cdot \boldsymbol{\xi} - \frac{1}{r}\frac{d}{dr}(r\xi_r) = \frac{im}{r}\xi_\theta + ik\xi_z, \tag{6b}$$

$$\zeta = i(\boldsymbol{\xi} \times \mathbf{B})_r = i\xi_\theta B_z - i\xi_z B_\theta. \tag{6c}$$

(The special case $m = k = 0$ will be treated later on.) The cylindrical components ξ_θ and ξ_z are given in terms of η and ζ by

$$\xi_\theta = -i\frac{kr\zeta + rB_\theta\eta}{krB_z + mB_\theta}; \qquad \xi_z = i\,\frac{m\zeta - rB_z\eta}{krB_z + mB_\theta}. \tag{7}$$

We substitute these expressions for ξ_θ and ξ_z in the energy integral as given by Eq. (3), using only the real part of the complex $\boldsymbol{\xi}$ because $W(\boldsymbol{\xi})$ is a quadratic rather than a linear form, and eliminating $\mathbf{J}$ and ∇P with the help of Eqs. (1) and (2). The result, after some tedious but straightforward algebra, is

$$W(\xi,\eta,\zeta) = \frac{\pi}{2}\int_a^b r\,dr\left\{\Lambda\left(\xi,\frac{d\xi}{dr}\right) + \gamma P\left[\eta + \frac{1}{r}\frac{d}{dr}(r\xi)\right]^2 + \frac{k^2r^2+m^2}{r^2}\left[\zeta - \zeta_0\left(\xi,\frac{d\xi}{dr}\right)\right]^2\right\}, \tag{8}$$

where

$$\Lambda\left(\xi,\frac{d\xi}{dr}\right) = \frac{1}{k^2r^2+m^2}\left[(krB_z + mB_\theta)\frac{d\xi}{dr} + (krB_z - mB_\theta)\frac{\xi}{r}\right]^2 + \left[(krB_z + mB_\theta)^2 - 2B_\theta\frac{d}{dr}(rB_\theta)\right]\frac{\xi^2}{r^2}, \tag{9}$$

$$\zeta_0\left(\xi,\frac{d\xi}{dr}\right) = \frac{r}{k^2r^2+m^2}\left[(krB_\theta - mB_z)\frac{d\xi}{dr} - (krB_\theta + mB_z)\frac{\xi}{r}\right]. \tag{10}$$

As Rosenbluth has pointed out (*24*), one can now obtain a very simple sufficient condition for the stability of a tubular pinch. Since every term in the energy integral is positive-definite except possibly the term in Λ containing $B_\theta d(rB_\theta)/dr$, a tubular pinch is stable for all m and k if $|rB_\theta|$ is an everywhere-decreasing function of r. (This condition, of course, could never be satisfied in a columnar pinch.[11])

[11] Since Suydam's condition is necessary for stability, it should obviously be a consequence of Rosenbluth's condition. That this is so can easily be verified by transforming the inequality (5) into

$$rB_z^2[d\log(\mu r^4)/dr]^2 > 8B^2\, d\log(rB_\theta)/dr.$$

A pinch is stable for specified values of m and k if and only if $W(\xi,\eta,\zeta)$ is positive for every set of trial functions (ξ,η,ζ) with ξ satisfying the boundary conditions[12]

$$\xi(a) = \xi(b) = 0. \tag{11}$$

Let us define

$$W(\xi) = \min_{\eta,\zeta} W(\xi,\eta,\zeta). \tag{12}$$

Then $W(\xi,\eta,\zeta)$ is positive for every (ξ,η,ζ) if and only if $W(\xi)$ is positive for every ξ.

The indicated minimization with respect to η and ζ is trivial. Since the terms involving η and ζ are positive-definite and can be made to vanish by setting

$$\eta = -\frac{1}{r}\frac{d}{dr}(r\xi), \tag{13a}$$

$$\zeta = \zeta_0\left(\xi, \frac{d\xi}{dr}\right), \tag{13b}$$

we obtain

$$W(\xi) = \frac{\pi}{2}\int_a^b r\,dr\,\Lambda\left(\xi, \frac{d\xi}{dr}\right). \tag{14}$$

Equation (13a) is equivalent to $\nabla\cdot\xi = 0$; hence W is minimized by incompressible displacements. Furthermore, Λ does not contain γ, so that the stability criterion is independent of γ. (Other features of the motion do, however, depend on γ.) In particular, by choosing $\gamma = \infty$ we see that the stability criterion is unchanged by replacing the plasma with an incompressible fluid.[13]

A more convenient form of the energy principle is obtained by integrating Eq. (14) by parts to eliminate the term in $\xi\,d\xi/dr$:

$$W(\xi) = \frac{\pi}{2}\int_a^b dr\left[f\left(\frac{d\xi}{dr}\right)^2 + g\xi^2\right], \tag{15}$$

[12] The inner boundary condition must be modified slightly in the case of a columnar pinch (see Section V).

[13] This result has also been obtained on the basis of a marginal stability argument by Shafranov for the special case of a diffuse pinch with a uniform current distribution (*23*) and by Dungey and Loughhead for a sharp pinch (*25*). Tayler has raised certain objections to the marginal stability argument (*26*); although his objections are justified, they do not apply to our derivation, which is based on the energy principle.

where f and g, functions of r with m and k as parameters, are given by

$$f = \frac{r(krB_z + mB_\theta)^2}{k^2r^2 + m^2}, \tag{16}$$

$$g = \frac{1}{r}\frac{(krB_z - mB_\theta)^2}{k^2r^2 + m^2} + \frac{1}{r}(krB_z + mB_\theta)^2 - \frac{2B_\theta}{r}\frac{d}{dr}(rB_\theta) - \frac{d}{dr}\left(\frac{k^2r^2B_z^2 - m^2B_\theta^2}{k^2r^2 + m^2}\right). \tag{17}$$

The coefficient f is never negative, but g may have either sign. Another useful expression for g is obtained with the help of Eq. (2):

$$g = \frac{2k^2r^2}{k^2r^2 + m^2}\frac{dP}{dr} + \frac{1}{r}(krB_z + mB_\theta)^2\frac{k^2r^2 + m^2 - 1}{k^2r^2 + m^2} + \frac{2k^2r}{(k^2r^2 + m^2)^2}(k^2r^2B_z^2 - m^2B_\theta^2). \tag{18}$$

Excluding $m = 0$, let us compare displacements with different values of m but with the same value of $q = k/m$. If we make the substitution $k = mq$ in Eqs. (16) and (17), we find that the second term in Eq. (17) is the only one that depends on m. Since this term is positive-definite and proportional to m^2, the least stable displacements must be those with $m = \pm 1$. Consequently, if a linear pinch is stable for $m = 1$, $-\infty < k < \infty$, it is also stable for all higher values of m. (We are using the fact that $W(\xi)$ is unchanged when m and k both change sign.)

If $m = 0$, Eq. (15) reduces to

$$W(\xi) = W_0(\xi) + \frac{\pi k^2}{2}\int_a^b r\,dr\,B_z^2\xi^2, \tag{19}$$

where

$$W_0(\xi) = \frac{\pi}{2}\int_a^b dr\left[rB_z^2\left(\frac{d\xi}{dr}\right)^2 + \left(\frac{B_z^2}{r} + 2\frac{dP}{dr}\right)\xi^2\right]. \tag{20}$$

The term containing k^2 is positive-definite; hence the pinch is stable for all k if it is stable in the limit as $k \to 0$.

Finally, we must consider displacements for which m and k both vanish.[14] These displacements can be divided into two classes, those for which ξ_r vanishes identically and those for which ξ_θ and ξ_z vanish identically. We have $W(\xi) = 0$

[14] This is the exceptional case mentioned in footnote 6; for these displacements the hydromagnetic energy principle does not agree with the energy principle derived from the Boltzmann equation.

for displacements of the first class, which are therefore marginally stable. But since it is physically obvious that these displacements are not only marginally but neutrally stable, we can forget about them. Displacements of the second class involve radial expansions and compressions of the plasma, so that $\nabla \cdot \xi$ cannot vanish. Equation (15) is therefore not applicable, and we may expect the stability criterion for this case to differ from the criterion for $m = 0, k \to 0$. By direct substitution in Eq. (3) we obtain, after an integration by parts,

$$W = \frac{\pi}{2} \int_a^b dr \left[r(B_z^2 + B_\theta^2 + \gamma P) \left(\frac{d\xi}{dr}\right)^2 + 2(B_z^2 + \gamma P)\xi \frac{d\xi}{dr} + (B_z^2 - B_\theta^2 + \gamma P) \frac{\xi^2}{r} \right], \tag{21}$$

where ξ is again an abbreviation for ξ_r. (Since ξ_θ and ξ_z do not enter, there is nothing analogous to the minimization with respect to η and ζ.) Let us subtract Eq. (20) from Eq. (21). Then, using Eq. (2) and integrating by parts, we obtain

$$W - W_0 = \frac{\pi}{2} \int_a^b r\, dr \left[B_\theta^2 \left(\frac{d\xi}{dr} - \frac{\xi}{r}\right)^2 + \gamma P \left(\frac{d\xi}{dr} + \frac{\xi}{r}\right)^2 \right], \tag{22}$$

which is necessarily positive. Consequently, if the pinch is stable for $k \to 0$, it is also stable for $k = 0$.[15]

The results of the last three paragraphs are summarized in the following theorem.[16]

THEOREM 1. A linear pinch is stable for all values of m and k if and only if it is stable for $m = 1, -\infty < k < \infty$ and for $m = 0, k \to 0$.

This theorem obviously allows a considerable simplification in the testing of a pinch for stability. If we wish to find out whether a specified pinch is stable for all values of m and k, we need only examine the special cases mentioned in the theorem. It will sometimes be of interest, however, if a pinch is known to be unstable, to find out exactly which values of m and k give rise to the instability. For this reason we shall continue to state our results in terms of a general m without restricting ourselves to $m = 0$ and $m = 1$.

III. PROPERTIES OF THE EULER-LAGRANGE SOLUTIONS

We have now reduced the energy integral $W(\xi)$ to the simple form given by Eq. (15). If we try to minimize this form with respect to ξ, we are led to the

[15] This conclusion is not affected by the discrepancy between the two energy principles, since Kruskal and Oberman have shown (7) that hydromagnetic stability is always a sufficient condition for stability according to the Boltzmann equation.

[16] This theorem, in a somewhat weaker form, was first proved by Rosenbluth for the special case of a sharp columnar pinch (14); this is the paper in which attention was restricted to displacements that are continuous across the current layer.

Euler–Lagrange equation

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0, \tag{23}$$

the solutions of which give stationary, although perhaps not minimal, values for $W(\xi)$. This equation has a singular point wherever f vanishes, and we can see from Eq. (16) that this happens wherever

$$krB_z + mB_\theta = 0, \tag{24}$$

or in terms of the quantity μ entering into Suydam's condition, wherever $k + m\mu = 0$. (The singular points can also be characterized as those values of r for which $m\theta + kz$, the phase of the displacement, is constant along a line of force.[17]) For given values of m and k, let $r_s^{(1)}, r_s^{(2)}, \cdots, r_s^{(q)}$ be the singular points. (The number and location of the singular points are, of course, functions of m and k.) These points divide the interval $a < r < b$ into $q + 1$ subintervals $a < r < r_s^{(1)}$, $r_s^{(1)} < r < r_s^{(2)}, \cdots, r_s^{(q)} < r < b$, which we shall call the independent subintervals.

In general it is not possible to continue an Euler–Lagrange solution past a singular point.[18] For this reason we cannot speak of an Euler–Lagrange solution in the entire interval $a < r < b$; each solution is defined in only one of the independent subintervals. We shall see in the next section, however, that the energy integral can be minimized in each of these subintervals separately, so that there will never be any need to continue a solution past a singular point.

As is now evident, we shall have occasion to evaluate the energy integral not only over the full interval $a < r < b$, but also over subintervals $r_1 < r < r_2$, where $a < r_1 < r_2 < b$. We therefore introduce the notation

$$W(r_1, r_2; \xi) = \frac{\pi}{2}\int_{r_1}^{r_2} dr\left[f\left(\frac{d\xi}{dr}\right)^2 + g\xi^2\right]. \tag{25}$$

The displacement ξ in Eq. (25) may be an Euler–Lagrange solution if there are no singular points between r_1 and r_2, in which case W is especially easy to evaluate:

$$\begin{aligned} W(r_1, r_2; \xi) &= \frac{\pi}{2}\int_{r_1}^{r_2} dr\left[f\left(\frac{d\xi}{dr}\right)^2 + \frac{d}{dr}\left(f\frac{d\xi}{dr}\right)\xi\right] \\ &= \frac{\pi}{2}\int_{r_1}^{r_2} dr\,\frac{d}{dr}\left(f\xi\frac{d\xi}{dr}\right) = \frac{\pi}{2} f\xi\frac{d\xi}{dr}\bigg|_{r_1}^{r_2}. \end{aligned} \tag{26}$$

[17] In a columnar pinch the point $r = 0$ is also singular. We shall consider the effect of this singular point in Section V, restricting ourselves in the meantime to singular points that satisfy Eq. (24).

[18] This is true only because ξ is a real variable. If it were complex we could go around the singular point by analytic continuation, but the resulting solutions would generally be multivalued.

In particular, if r_1 and r_2 are vanishing points of $f\xi$, the energy integral is equal to zero.

Let us now examine the behavior of the Euler–Lagrange solutions in the neighborhood of a singular point r_s. (We are, of course, interested only in one-sided neighborhoods of r_s, for we have seen that an Euler–Lagrange solution is defined in only one of the independent subintervals.) If $x = |r - r_s|$, the coefficients f and g are approximated in the neighborhood of r_s by

$$f = \alpha x^2, \quad \text{with} \quad \alpha > 0, \tag{27}$$

and

$$g = \beta, \tag{28}$$

where α and β are independent of x, and the Euler–Lagrange equation reduces to

$$\alpha \frac{d}{dx}\left(x^2 \frac{d\xi}{dx}\right) - \beta\xi = 0. \tag{29}$$

The solutions are x^{-n_1} and x^{-n_2}, where n_1 and n_2 are the roots of the indicial equation

$$n^2 - n - (\beta/\alpha) = 0. \tag{30}$$

(We are neglecting the marginal case where $n_1 = n_2$.)

The condition for n_1 and n_2 to be real and unequal is that $\alpha + 4\beta$ should be positive. If $\alpha + 4\beta$ is negative, the roots of the indicial equation are complex conjugates, and the real solutions of the Euler–Lagrange equation are

$$\xi = x^{-n} + x^{-n^*} \quad \text{and} \quad \xi = i(x^{-n} - x^{-n^*}). \tag{31}$$

These solutions are oscillatory in the neighborhood of r_s.

Using the fact that $k + m\mu = 0$ at a singular point, we can obtain the expressions for α and β from Eqs. (16) and (18):

$$\begin{aligned} \alpha &= \frac{r_s}{k^2 r_s^2 + m^2}\left(kr\frac{dB_z}{dr} + kB_z + m\frac{dB_\theta}{dr}\right)^2_{r=r_s} \\ &= \frac{r_s B_\theta^2 B_z^2}{B^2}\left(\frac{d\log\mu}{dr}\right)^2, \end{aligned} \tag{32a}$$

$$\beta = \frac{2B_\theta^2}{B^2}\frac{dP}{dr}, \tag{32b}$$

where $B^2 = B_\theta^2 + B_z^2$. The condition for nonoscillatory solutions, $\alpha + 4\beta > 0$, then reduces to Suydam's condition:

$$\frac{r}{8}B_z^2\left(\frac{d\log\mu}{dr}\right)^2 + \frac{dP}{dr} > 0, \tag{33}$$

where all quantities are evaluated at r_s. We shall prove later on that a pinch with oscillatory solutions is unstable, thus verifying that Suydam's condition is necessary for stability.

Let us suppose that Suydam's condition is fulfilled at r_s, so that n_1 and n_2 are real. It follows from Eq. (30), the indicial equation, that $n_1 + n_2 = 1$, and we shall take $n_2 > n_1$. The solution x^{-n_2} is necessarily infinite at r_s, but x^{-n_1} can be either infinite or zero, depending on whether n_1 is positive or negative. Furthermore, $W(\xi)$ converges at r_s for the solution x^{-n_1} but diverges for x^{-n_2}. We shall say that an Euler–Lagrange solution is small at r_s if it behaves like a constant multiple of x^{-n_1} in the neighborhood of r_s. Now let $\xi_s(r)$ be an Euler–Lagrange solution that is small at r_s, and let $\xi(r;r_1)$ be one that vanishes at the nonsingular point r_1. These solutions are uniquely determined except for normalization factors, and it is easily seen that with a proper choice of the normalization factors we have

$$\xi_s(r) = \lim_{r_1 \to r_s} \xi(r;r_1). \tag{34}$$

Thus the smallness of a solution at a singular point is analogous to the vanishing of a solution at a nonsingular point.

We shall find that the stability criteria depend critically on the existence or nonexistence of points at which certain Euler–Lagrange solutions vanish. The following theorem, a special case of Sturm's separation theorem (*27*), will then be of interest.

THEOREM 2. If $\xi_a(r)$ and $\xi_b(r)$ are any two linearly independent solutions of the Euler–Lagrange equation in the same independent subinterval I, and if $\xi_a(r)$ vanishes at r_1 and r_2, two distinct interior points of I, then $\xi_b(r)$ vanishes at some point between r_1 and r_2. (We shall assume that r_1 and r_2 are consecutive vanishing points of ξ_a; it is clear that this involves no loss in generality.)

PROOF. Since $\xi_a(r)$ and $\xi_b(r)$ are solutions of the Euler–Lagrange equation, we have

$$(f\xi_a')' - g\xi_a = 0, \tag{35}$$

$$(f\xi_b')' - g\xi_b = 0. \tag{36}$$

Multiplying Eqs. (35) and (36) by ξ_b and ξ_a respectively, and then subtracting, we obtain

$$fw' + f'w = 0, \tag{37}$$

where $w = \xi_b\xi_a' - \xi_a\xi_b'$, the Wronskian of the two solutions. It follows that wf is constant, so that w must have the same sign everywhere. We have $w(r_1) = \xi_b(r_1)\xi_a'(r_1)$ and $w(r_2) = \xi_b(r_2)\xi_a'(r_2)$. Clearly, $\xi_a'(r_1)$ and $\xi_a'(r_2)$ have opposite signs; hence $\xi_b(r_1)$ and $\xi_b(r_2)$ also have opposite signs, and $\xi_b(r)$ must vanish somewhere between r_1 and r_2.

Corollary 2-1. Let I be an independent subinterval with a singular endpoint r_s, suppose that Suydam's condition is fulfilled at r_s, and let $\xi_s(r)$ be an Euler–Lagrange solution that is small at r_s. Then, if $\xi_s(r_2) = 0$, where r_2 is an interior point of I, every other solution vanishes at some point between r_s and r_2.

Proof. If r_1 is an interior point of I that is close to r_s, it follows from Eq. (34) that $\xi(r;r_1)$ vanishes not only at r_1 but also at some point close to r_2. The corollary is then obtained as a limiting case of the theorem if $\xi(r;r_1)$ is used in the place of $\xi_a(r)$.

IV. THE MINIMIZATION OF $W(\xi)$

We must find out whether $W(\xi)$ is positive for every physically admissible trial function $\xi(r)$. These functions, in addition to satisfying the boundary conditions (11), must all be continuous, since they represent displacements of a real fluid, the plasma. We can, however, extend the class of admissible trial functions by allowing finite jumps at the singular points. To see this, consider a $\xi(r)$ that varies rapidly in the neighborhood of a singular point r_s. More precisely, suppose that ξ changes by a finite amount in the interval from $r_s - \eta$ to $r_s + \eta$, where η is very small. In this interval f and $d\xi/dr$ are of order η^2 and η^{-1}, respectively, so that $W(r_s - \eta, r_s + \eta;\xi)$ is of order η. If we take the limit as $\eta \to 0$, then ξ becomes discontinuous and $W(a,b;\xi)$ approaches a finite limit. This means that we can allow displacements with finite jumps at the singular points, since displacements of this type can be approximated to any desired degree of accuracy by continuous displacements. (We cannot, of course, allow finite jumps at nonsingular points; they would make infinite contributions to W.)

We shall say that a pinch is stable in an independent subinterval I if the energy integral is positive over I for every ξ. We then have the following theorem:

Theorem 3. A linear pinch is stable for specified values of m and k if and only if it is stable in each of the independent subintervals.

Proof. Let r_s be a singular point, and write $W = W_a + W_b$, where W_a and W_b are the contributions from the intervals $a < r < r_s$ and $r_s < r < b$ respectively. (It is clearly sufficient to consider only one of the singular points.) We shall prove that W is positive for every ξ if and only if W_a and W_b are individually positive for every ξ. The "if" part of this statement is obvious, and to prove the "only if" part, suppose that a ξ exists for which W_a is negative. Then define ξ_1 as follows

$$\xi_1(r) = \begin{matrix} \xi(r), & a < r < r_s \\ 0, & r_s < r < b \end{matrix} . \tag{38}$$

(The displacement ξ_1 will in general have a finite jump, but we have seen that this is permissible at a singular point.) We now have

$$W(a, b; \xi_1) = W(a, r_s ; \xi_1) = W(a, r_s ; \xi) < 0, \tag{39}$$

which is what we set out to prove.

This theorem enables us to minimize $W(\xi)$ for the complete interval $a < r < b$ by minimizing it in each of the independent subintervals separately. We are not yet prepared, however, to minimize $W(\xi)$ for an interval with singular endpoints. Let us therefore consider the following preliminary minimization problem: If r_1 and r_2 are nonsingular points of the same independent subinterval, find the displacement $\xi(r)$ that minimizes $W(r_1, r_2 ; \xi)$ while satisfying the boundary conditions

$$\begin{aligned} \xi(r_1) &= c_1 , \\ \xi(r_2) &= c_2 . \end{aligned} \tag{40}$$

Since the minimizing ξ, if there is one, will make W stationary, we know that it must be an Euler–Lagrange solution. Let $\xi_1(r)$ and $\xi_2(r)$ be the Euler–Lagrange solutions satisfying

$$\xi_1(r_1) = 0, \qquad \xi_1'(r_1) = 1, \tag{41}$$

$$\xi_2(r_1) = 1, \qquad \xi_2'(r_1) = 0. \tag{42}$$

Now if $\xi_1(r_2) \neq 0$, there will exist a unique Euler–Lagrange solution $\xi_0(r)$ satisfying the boundary conditions (40); it is given by

$$\xi_0(r) = \frac{c_2 - c_1\xi_2(r_2)}{\xi_1(r_2)}\, \xi_1(r) + c_1\xi_2(r). \tag{43}$$

We shall prove that $W(r_1, r_2 ; \xi)$ is minimized by $\xi_0(r)$ if $\xi_1(r)$ never vanishes in the interval $r_1 < r < r_2$.[19] As usual, it does not matter what happens in the marginal case where $\xi_1(r)$ vanishes at r_2, and we shall prove that W is not minimized by $\xi_0(r)$ when $\xi_1(r)$ vanishes anywhere between r_1 and r_2.

Suppose first that $\xi_1(r)$ never vanishes in the interval $r_1 < r \leqq r_2$. To prove that $\xi_0(r)$ minimizes the energy integral we shall define an auxiliary integral $W\dagger(r_1, r_2 ; \xi)$, called the Hilbert invariant integral,[20] with the following properties: $W\dagger(\xi)$ has the same value for all ξ satisfying (40), $W\dagger(\xi) < W(\xi)$ for every such ξ other than ξ_0, and $W\dagger(\xi_0) = W(\xi_0)$. It will then follow that the inequality

$$W(\xi) \geqq W\dagger(\xi) = W\dagger(\xi_0) = W(\xi_0) \tag{44}$$

[19] This condition for minimization is a special case of Jacobi's condition (*28*).

[20] The use of a Hilbert integral is a standard technique in the calculus of variations. An elementary exposition of this technique is given in Ref. *28*.

holds for every ξ that satisfies (40), and this will prove that W is indeed minimized by ξ_0.

Before defining the Hilbert integral, let us consider the singly infinite family of functions

$$\xi_A(r) = A\xi_1(r) + c_1\xi_2(r), \tag{45}$$

where A is an arbitrary constant. These functions are obviously solutions of the Euler–Lagrange equation, and ξ_0 is one of them. Their graphs, which we shall call characteristic curves, cover the strip $r_1 < r \leqq r_2$ in the r,ξ plane (see Fig. 1). This is true only because $\xi_1(r)$ never vanishes in the interval $r_1 < r \leqq r_2$; if it did vanish at some point r_0 of that interval, then the characteristic curves would all intersect at the point $[r_0, c_1\xi_2(r_0)]$, and none of them would pass through any other point on the line $r = r_0$ parallel to the ξ axis (see Fig. 2).

We now define $p(r,\xi)$, a function of position in the r,ξ plane, as the slope of the characteristic curve passing through the point (r,ξ). This function is defined everywhere in the strip $r_1 < r \leqq r_2$. On the line $r = r_1$, as is evident from Fig. 1, we have $p \to \pm\infty$, except at the point (r_1, c_1), where p takes on finite values depending on the direction of approach.

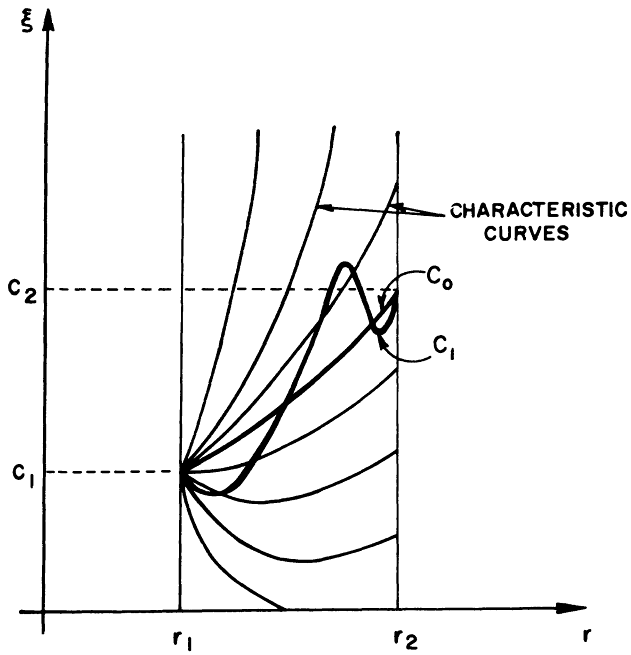

FIG. 1. Illustration of Theorem 4. The curve C_0 is the graph of $\xi_0(r)$, and C_1 is the graph of any other function $\xi(r)$ with the same values at the endpoints.

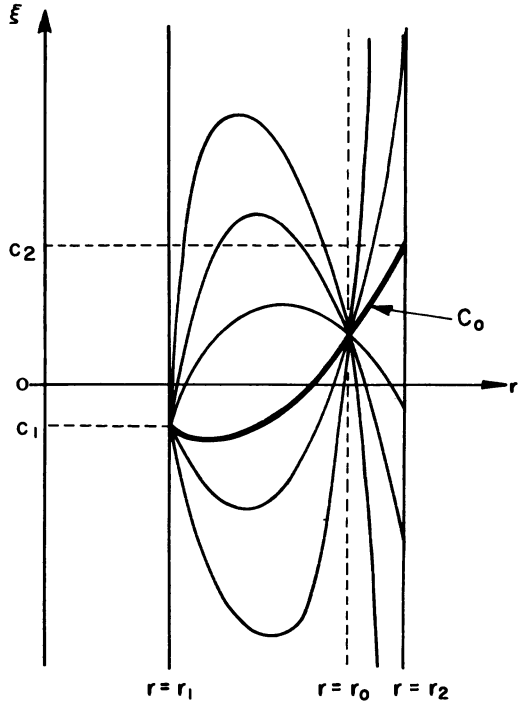

FIG. 2. Characteristic curves for an interval failing to satisfy the condition for minimization. The curve C_0 is the graph of $\xi_0(r)$.

Every characteristic curve satisfies the differential equations

$$d\xi = p\, dr, \tag{46}$$

$$d(fp) - g\xi\, dr = 0, \tag{47}$$

of which the first expresses the definition of p and the second the fact that the characteristic curves represent Euler–Lagrange solutions. Also, since p is a function of r and ξ, we have

$$dp = (\partial p/\partial r)\, dr + (\partial p/\partial \xi)\, d\xi. \tag{48}$$

Eliminating the differentials from Eqs. (46–48), we obtain the following useful relation between the partial derivatives of p:

$$(\partial/\partial r)(fp) + fp(\partial p/\partial \xi) - g\xi = 0. \tag{49}$$

The Hilbert integral can now be defined by

$$W\dagger(C) = \frac{\pi}{2}\int_C [(fp^2 + g\xi^2)\, dr + 2fp(d\xi - p\, dr)], \tag{50}$$

where C is a curve lying in the part of the r,ξ plane bounded by the lines $r = r_1$ and $r = r_2$. This definition, of course, can be applied only to curves for which the integral converges. For this reason the left endpoint of C, if it is on the line $r = r_1$, must be the point (r_1, c_1). But this condition, since it is satisfied by the graph of any function $\xi(r)$ satisfying the boundary conditions (40), will not cause any trouble. It is also satisfied by the characteristic curves, for which $W\dagger$ is obviously equal to W.

Let us write the Hilbert integral in the form

$$W\dagger(C) = \frac{\pi}{2}\int_C (F_r\, dr + F_\xi\, d\xi), \tag{51}$$

where

$$F_r = g\xi^2 - fp^2, \qquad F_\xi = 2fp. \tag{52}$$

From Eqs. (49) and (52) we obtain

$$(\partial F_r/\partial\xi) - (\partial F_\xi/\partial r) = 0. \tag{53}$$

The Hilbert integral is therefore path-independent; its value depends only on the endpoints of C.

Let C_0 be the graph of $\xi_0(r)$, and let C_1 be the graph of any other function $\xi(r)$ satisfying the boundary conditions (40). We have $W\dagger(C_1) = W\dagger(C_0)$ because $W\dagger$ is path-independent, and $W\dagger(C_0) = W(C_0)$ because C_0 is a characteristic curve. Furthermore,

$$\begin{aligned} W(C_1) - W\dagger(C_1) &= \frac{\pi}{2}\int_{C_1}\left\{\left[f\left(\frac{d\xi}{dr}\right)^2 + g\xi^2\right] dr \right. \\ &\qquad \left. - (fp^2 + g\xi^2)\, dr - 2fp(d\xi - p\, dr)\right\} \\ &= \frac{\pi}{2}\int_{C_1} f\left(\frac{d\xi}{dr} - p\right)^2 dr, \end{aligned} \tag{54}$$

which is always positive because f is always positive. As we have already seen, it follows immediately that $W(C_1) > W(C_0)$, which proves that ξ_0 is the minimizing displacement. We state our result as a theorem:

THEOREM 4. If r_1 and r_2 are nonsingular points of the same independent subinterval, and if the nontrivial Euler–Lagrange solutions that vanish at r_1 never vanish in the interval $r_1 < r \leqq r_2$,[21] then the energy integral $W(r_1, r_2; \xi)$ is smaller for an Euler–Lagrange solution $\xi_0(r)$ than it is for any other $\xi(r)$ with the same boundary values.

[21] Since these solutions are constant multiples of each other, this condition will hold either for all of them or for none.

Corollary 4-1. Under the conditions of Theorem 4, $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that satisfies the boundary conditions $\xi(r_1) = \xi(r_2) = 0$.

Proof. By Theorem 4, we have $W(\xi) > W(\xi_0)$, and $W(\xi_0) = 0$ because ξ_0, the Euler–Lagrange solution that vanishes at both endpoints, must vanish identically.

Theorem 4 states that W is locally minimized by the Euler–Lagrange solutions; i.e., it is minimized whenever r_1 and r_2 are sufficiently close together. Specifically, they are sufficiently close together when r_2 is to the left of the next vanishing point, if one exists, of the nontrivial Euler–Lagrange solutions that vanish at r_1. We shall see below that this condition on the closeness of r_1 and r_2 is, if the marginal case is neglected, not only sufficient for minimization but also necessary.

The endpoints r_1 and r_2 do not enter symmetrically into the statement of Theorem 4 and its corollary. Nevertheless, their roles can be interchanged because of Theorem 2. We can also obtain from Theorem 2 a symmetrical form of the criterion of closeness for r_1 and r_2 : r_1 and r_2 are sufficiently close for W to be minimized by ξ_0 if there is no nontrivial solution of the Euler–Lagrange equation that vanishes at two distinct points between r_1 and r_2. We shall, however, continue to state our results in a form that is unsymmetrical with respect to the two endpoints; the roles of the endpoints can always be interchanged by an application of Theorem 2 or its corollary.

Now suppose that $\xi_1(r)$, the Euler–Lagrange solution satisfying Eqs. (41), vanishes at some point r_0 between r_1 and r_2. The characteristic curves for this case are shown in Fig. 2. It is clear, first of all, that p is infinite on the line $r = r_0$, so that the proof of Theorem 4 does not apply here. In the place of Theorem 4 we have

Theorem 5. If r_1 and r_2 are nonsingular points of the same independent subinterval, and if the nontrivial Euler–Lagrange solutions that vanish at r_1 also vanish at some point r_0 between r_1 and r_2, then for any Euler–Lagrange solution $\xi_0(r)$ there exist functions $\xi(r)$ with the same boundary values and with $W(r_1, r_2; \xi) < W(r_1, r_2; \xi_0)$.

Proof. Let the Euler–Lagrange solution $\xi_0(r)$ be represented by the curve (1 2 3 4 5) in Fig. 3, let the arc (1 6 7 3) be another characteristic curve between r_1 and r_0, and let $\xi_1(r)$ be the function represented by the broken curve (1 6 7 3 4 5). The Hilbert integral has the same properties as before if it is restricted to curves lying entirely within the strip $r_1 \leqq r \leqq r_0$. Applying it to the characteristic curves (1 2 3) and (1 6 7 3), we obtain $W(1\,6\,7\,3) = W^\dagger(1\,6\,7\,3) = W^\dagger(1\,2\,3) = W(1\,2\,3)$, so that $W(r_1, r_2; \xi_1) = W(r_1, r_2; \xi_0)$. Now replace the segment (7 3 4) with the arc (7 8 4) representing an Euler–Lagrange solution. Because of the local minimization property of Euler–Lagrange solutions, we can take the points 7 and 4 so close together that $W(7\,8\,4) < W(7\,3\,4)$.

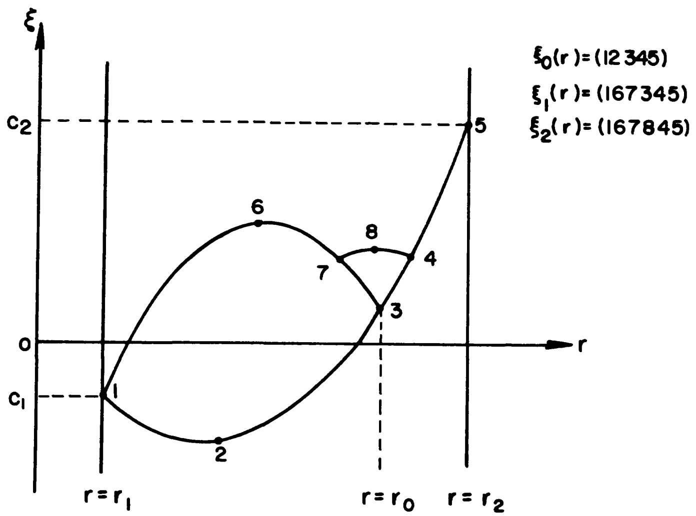

FIG. 3. Trial functions used in the proof of Theorem 5

Let $\xi_2(r)$ be represented by the curve (1 6 7 8 4 5). Then $W(r_1, r_2; \xi_2) < W(r_1, r_2; \xi_1) = W(r_1, r_2; \xi_0)$, which means that W is not minimized by $\xi_0(r)$.

The following corollary shows that the pinch is unstable whenever the Euler–Lagrange solutions fail to minimize the energy integral.

COROLLARY 5-1. Under the conditions of Theorem 5, there exists a $\xi(r)$ that makes $W(r_1, r_2; \xi)$ negative and satisfies the boundary conditions $\xi(r_1) = \xi(r_2) = 0$.

PROOF. As in Corollary 4-1, the Euler–Lagrange solution $\xi_0(r)$ that vanishes at r_1 and r_2 vanishes identically. Hence $W(r_1, r_2; \xi_0) = 0$, and according to Theorem 5 there exists a $\xi(r)$ vanishing at r_1 and r_2 for which W takes on a smaller value. (A displacement of this type is shown in Fig. 4.)

COROLLARY 5-2. Under the conditions of Theorem 5, the minimum value of $W(r_1, r_2; \xi)$ with respect to the class of displacements satisfying the boundary conditions (40) is minus infinity.

PROOF. By hypothesis, the Euler–Lagrange solutions that vanish at r_1 also vanish at some point r_0 between r_1 and r_2. Let us pick a point r_3 between r_0 and r_2 (see Fig. 5). Then, according to Theorem 2, the Euler–Lagrange solutions that vanish at r_3 also vanish at some point r_4 between r_1 and r_0. Theorem 5 can then be applied to any interval $r_4 < r < r_5$, where r_5 is between r_3 and r_2, and Corollary 5-1 guarantees the existence of a displacement $\xi_1(r)$ that vanishes outside this interval and makes the energy integral negative. Now let $\xi_2(r)$ be any displacement that satisfies the boundary conditions (40) and vanishes between

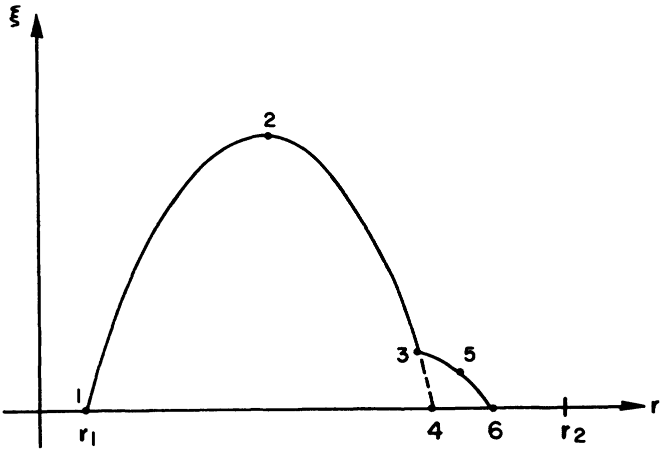

FIG. 4. Illustration of Corollary 5-1. The segments (1234) and (356) represent Euler–Lagrange solutions, and $\xi(r)$ = (12356) is a displacement for which $W(r_1, r_2; \xi)$ is negative.

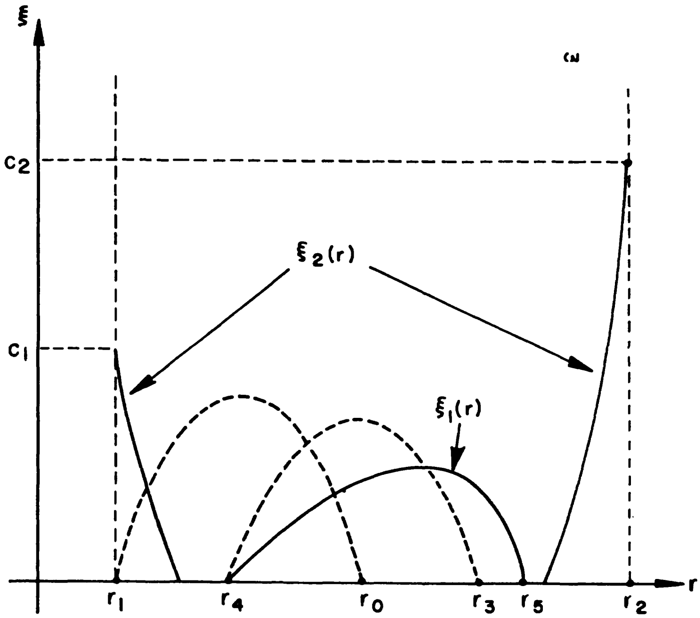

FIG. 5. Illustration of Corollary 5-2. The dotted curves represent Euler–Lagrange solutions, and the solid curves represent the displacements $\xi_1(r)$ and $\xi_2(r)$ mentioned in the text.

r_4 and r_5, and let $\xi_A(r) = A\xi_1(r) + \xi_2(r)$. Since the functions $\xi_1(r)$ and $\xi_2(r)$ do not overlap, we have

$$W(r_1, r_2; \xi_A) = A^2 W(r_1, r_2; \xi_1) + W(r_1, r_2; \xi_2). \tag{55}$$

The displacement $\xi_A(r)$ satisfies the boundary conditions (40), and it follows from Eq. (55) that the energy integral is negatively infinite in the limit as $A \to \infty$.

Let us now define a regular singular point as a singular point at which Suydam's condition is fulfilled, so that small solutions exist. The next two theorems are the analogs of Theorem 4 for intervals with singular endpoints.

THEOREM 6. Let r_1 be a regular singular point and r_2 a nonsingular point of one of the two independent subintervals adjacent to r_1, say the one extending to the right. If the nontrivial Euler–Lagrange solutions that are small at r_1 never vanish in the interval $r_1 < r \leqq r_2$ (see footnote 21), and if $\xi_0(r)$ is one of these solutions, then $W(r_1, r_2; \xi)$ has a smaller value for $\xi_0(r)$ than it does for any function $\xi(r)$ satisfying the conditions

$$\xi(r) \text{ bounded}, \qquad r_1 \leqq r \leqq r_2, \tag{56a}$$

$$\xi(r_2) = \xi_0(r_2), \tag{56b}$$

with the exception of ξ_0 itself if $n_1 < 0$, since ξ_0 satisfies the conditions (56) in that case. (Notice that there is no restriction on the value of ξ at the singular point r_1, except that it should be finite.)

PROOF. Let us make use of the Euler-Lagrange solutions that are small at r_1 to define a family of characteristic curves as in the proof of Theorem 4 (see Fig. 6). These solutions are given in the neighborhood of r_1 by

$$\xi = \xi_A(r) \cong A(r - r_1)^{-n_1}, \tag{57}$$

where A is some constant; hence

$$\begin{aligned} p(r,\xi) = \xi_A'(r) &\cong -An_1(r - r_1)^{-n_1-1} \\ &\cong -n_1\xi/(r - r_1). \end{aligned} \tag{58}$$

Thus p becomes infinite as $r \to r_1$, but the integrand in the Hilbert integral $W\dagger$ remains finite because f is proportional to $(r - r_1)^2$. Furthermore, there is no other value of r for which p becomes infinite, since the relevant Euler–Lagrange solutions never vanish between r_1 and r_2. It follows that $W\dagger(C)$ is well-defined for any curve C in the strip $r_1 \leqq r \leqq r_2$ of the r,ξ plane. As before, we can prove that $W\dagger(C)$ depends only on the endpoints of C, and that $W(C) \geqq W\dagger(C)$, the equality sign holding only when C is a characteristic curve. Now let $\xi_0(r_2) = c$, let C_1 be the graph of any function $\xi(r)$ satisfying the conditions (56), and let

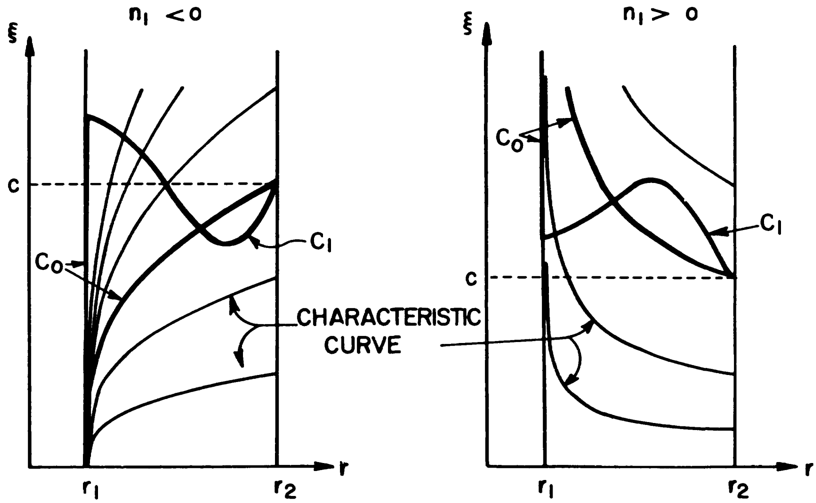

FIG. 6. Illustration of Theorem 6. The curve C_0 is the graph of $\xi_0(r)$ plus a segment of the line $r = r_1$ and C_1 is the graph of some function $\xi(r)$ satisfying the conditions (56).

C_0 be the graph of $\xi_0(r)$ plus a vertical segment joining the left endpoint of C_1 to the point (r_1, M), where

$$M = \begin{cases} 0 & \text{if } n_1 < 0, \\ +\infty & \text{if } n_1 > 0 \text{ and } c > 0, \\ -\infty & \text{if } n_1 > 0 \text{ and } c < 0. \end{cases} \tag{59}$$

It is easily verified that $W\dagger$ vanishes for the vertical segment. Then $W(r_1, r_2; \xi) = W(C_1) > W\dagger(C_1) = W\dagger(C_0) = W(r_1, r_2; \xi_0)$, as we set out to prove.

COROLLARY 6-1. Under the conditions of Theorem 6, $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that vanishes at r_2 and remains bounded between r_1 and r_2.

THEOREM 7. If r_1 and r_2 are successive singular points, both of them regular, and if the nontrivial Euler–Lagrange solutions that are small at r_1 never vanish in the interval $r_1 < r \leqq r_2$, then $W(r_1, r_2; \xi)$ is positive for every nontrivial $\xi(r)$ that remains bounded in that interval.

PROOF. The characteristic curves are again defined by the Euler–Lagrange solutions that are small at r_1. In the neighborhood of r_2 these solutions are linear combinations of x^{-n_1} and x^{-n_2}, but x^{-n_2} dominates. Therefore, by analogy with Eq. (58), $p(r,\xi)$ is approximately equal to $n_2\xi/(r_2 - r)$, and the Hilbert integral converges at r_2 as well as at r_1. The rest of the proof is the same as it was for Theorems 4 and 6, except that C_0 is now defined as follows: If $\xi(r_1) = c_1$ and

$\xi(r_2) = c_2$, then C_0 is composed of the segments from (r_1, c_1) to $(r_1, 0)$, from $(r_1, 0)$ to $(r_2, 0)$, and from $(r_2, 0)$ to (r_2, c_2). (The curve C_0 represents a degenerate Euler–Lagrange solution; an Euler–Lagrange solution that is finite at both of the singular endpoints must vanish identically between the endpoints.)

Theorem 7 and Corollaries 4-1 and 6-1 give conditions under which the minimum value of W is zero. This does not mean that the stability is marginal, however, because the minimum value is attained only when ξ vanishes identically; we have proved that W is positive for every nontrivial ξ.

The reader may wonder why it was necessary to prove each theorem separately; why not prove Theorem 7, for example, by a straightforward limiting process applied to Corollary 4-1? The reason is that such a limiting process would show, not that $W(\xi)$ is positive for every nontrivial ξ, but only that it is non-negative.[22] The limiting process does, however, lead directly to the analog of Theorem 5 for an interval with singular endpoints:

THEOREM 8. Suppose that r_1 is a singular point, and that r_2 is a point, either singular or nonsingular, of the independent subinterval bounded on the left by r_1. Then the minimum value of $W(r_1, r_2; \xi)$ for the class of displacements satisfying the boundary conditions (40) is minus infinity in each of the following cases: (1) r_1 is irregular, and (2) r_1 is regular, and the nontrivial Euler–Lagrange solutions that are small at r_1 vanish at some point between r_1 and r_2.

PROOF. In each of the cases mentioned there exist Euler–Lagrange solutions that vanish at least twice between r_1 and r_2. This follows in the first case from the oscillatory nature of the solutions in the neighborhood of r_1 and in the second from Corollary 2-1. We can therefore find a subinterval with nonsingular endpoints for which Theorem 5 is applicable, and the present theorem follows immediately.

To summarize the results of this section, let $W_{\min}(r_1, r_2; c_1, c_2)$ be the minimum value of $W(r_1, r_2; \xi)$ for the class of displacements satisfying the boundary conditions (40), let $\xi_0(r)$ be the Euler–Lagrange solution that satisfies the boundary conditions

$$\xi(r_i) = c_i \quad \text{if } r_i \text{ is a nonsingular point } (i = 1 \text{ or } 2), \tag{60a}$$

$$\xi(r) \text{ is small at } r_i \quad \text{if } r_i \text{ is a regular singular point,} \tag{60b}$$

and let $\xi_1(r)$ be any nontrivial Euler–Lagrange solution that satisfies the boundary condition

$$\xi(r_1) = 0 \quad \text{if } r_1 \text{ is a nonsingular point,} \tag{61a}$$

$$\xi(r) \text{ is small at } r_1 \quad \text{if } r_1 \text{ is a regular singular point.} \tag{61b}$$

[22] For the same reason, separate minimization theorems should be stated and proved for intervals with the singular endpoint $r = 0$ in a columnar pinch (see below, Section V).

(We note that the solutions $\xi_0(r)$ and $\xi_1(r)$ are undefined when r_1 is an irregular singular point, and that $\xi_0(r)$ vanishes identically when r_1 and r_2 are both singular.) It has been shown that

$$W_{\min}(r_1, r_2; c_1, c_2) = W(r_1, r_2; \xi_0) \tag{62}$$

if $\xi_1(r)$ never vanishes in the interval $r_1 < r \leqq r_2$. This minimum value (which is independent of c_i when r_i is singular) is attained only when $\xi(r) = \xi_0(r)$ if r_1 and r_2 are both nonsingular, and not at all if at least one of them is singular, since $\xi_0(r)$ then fails to satisfy the boundary conditions (40). Finally, we have shown that $W_{\min} = -\infty$ in every nonmarginal case not covered by Eq. (62), i.e., whenever r_1 is an irregular singular point, and whenever $\xi_1(r)$ vanishes somewhere between r_1 and r_2. Because of this last result, the derivation of stability criteria is not hindered by failure of the Euler–Lagrange solutions to minimize the energy integral—the pinch is obviously unstable whenever this happens.

V. STABILITY CRITERIA

The following theorem, an immediate consequence of Corollary 5-1, gives a necessary condition for stability.

THEOREM 9. For specified values of m and k, a linear pinch is unstable in an independent subinterval I if there exists a nontrivial Euler–Lagrange solution that vanishes at two distinct interior points of I.

COROLLARY 9-1. A linear pinch is stable (1) for specified values of m and k, or (2) for all values, only if Suydam's condition is fulfilled (1) at every singular point, or (2) everywhere.[23]

Parts (1) and (2) of this corollary follow respectively from the oscillatory character of the Euler–Lagrange solutions in the neighborhood of an irregular singular point and from the fact that every point is singular for some values of m and k. The next corollary is the limiting form of Theorem 9 obtained by an application of Corollary 2-1.

COROLLARY 9-2. For specified values of m and k, let I be an independent subinterval with a regular singular endpoint r_1. Then the pinch is unstable in I if the nontrivial Euler–Lagrange solutions that are small at r_1 vanish at any interior point of I.

We now have necessary conditions for stability, and also, for a tubular pinch, a simple sufficient condition (see Section II). Assuming that there is no more than one singular point, Rosenbluth (*21*) has given the following necessary and sufficient condition: If there is no singular point (Case 1), then the nontrivial

[23] Although Suydam's condition, as expressed by the inequality (33), refers to a point that is singular only for certain specified values of m and k, it does not involve those values explicitly. It is this circumstance that makes possible such a simple result concerning stability for all values of m and k.

Euler–Lagrange solutions that vanish at $r = a$ do not vanish anywhere in the interval $a < r < b$; and if there is a singular point, say at r_s (Case 2), then the nontrivial solutions that vanish at $r = a$, and the ones that vanish at $r = b$, do not vanish anywhere in the intervals $a < r < r_s$ and $r_s < r < b$ respectively. He derives this condition by means of a marginal-stability argument, which he states only for Case 1. Since it is not entirely clear how this argument is to be extended to Case 2, it seems desirable to give a more rigorous proof, which we shall now do. Also, we shall make use of the small solutions to take account of situations where there is more than one singular point.

The following theorem and its corollaries give the necessary and sufficient conditions for stability of a tubular pinch, the columnar pinch requiring separate treatment because of the singular point at $r = 0$.[24] To simplify the statement of these conditions, the definition of smallness is extended to nonsingular points as follows: An Euler–Lagrange solution $\xi(r)$ is small at a nonsingular point r_1 if $\xi(r_1) = 0$.

Theorem 10. For specified values of m and k, a tubular pinch is stable in an independent subinterval I if and only if (1) Suydam's condition is fulfilled at the left endpoint[25] if that point is singular (i.e., not equal to a), and (2) the Euler–Lagrange solutions that are small at the left endpoint never vanish in the interior of I. (We are, as usual, neglecting the marginal case where these solutions are small at the right endpoint of I.)

Proof. The "only if" part follows immediately from Theorem 9 and its corollaries. The "if" part, since every displacement must satisfy the boundary conditions (11), follows from Corollaries 2-1 and 6-1 if I is bounded on the left by a and on the right by a singular point, from Corollary 6-1 if it is bounded on the left by a singular point and on the right by b, from Theorem 7 if it is bounded by two singular points, and from Corollary 4-1 if there are no singular points, so that I is bounded by a and b. (Our use of Theorems 6 and 7 implies that we are restricting ourselves to trial functions that are bounded in I. This restriction is legitimate even though the Euler–Lagrange solutions are unbounded, for $\xi(r)$ represents a physical displacement of the plasma and must therefore be finite even at the singular points.)

Combining this result with Theorem 3, we obtain

Corollary 10-1. A tubular pinch is stable if and only if (1) Suydam's condition is fulfilled at every point, and (2) for all values of m and k, and for every independent subinterval I, the Euler–Lagrange solutions that are small at the left endpoint of I never vanish in the interior of I.

[24] The analysis of singular points given in Section III is not applicable at $r = 0$ because Eqs. (27) and (28) are not valid there.

[25] Although Suydam's condition is needed at the left endpoint to guarantee the existence of small solutions, it need not be stated explicitly for the right endpoint because it follows from (2). For the same reason, it will not be mentioned at all in Corollary 10-2.

According to Theorem 2 and its corollary, the left endpoint of I can be replaced in Theorem 10 and in Corollary 10-1 by the right endpoint. Also, the stability criterion can be given a symmetrical form:

COROLLARY 10-2. A tubular pinch is stable for specified values of m and k if and only if there is no nontrivial Euler–Lagrange solution that vanishes twice in the same independent subinterval. Thus the stability condition given by Theorem 9 is not only necessary but also sufficient.

As we have already stated, a columnar pinch differs from a tubular pinch in certain respects. First, the left-hand boundary condition is not $\xi(0) = 0$ but

$$\xi(0) = 0 \quad \text{if} \quad m \neq \pm 1, \tag{63a}$$

$$\xi'(0) = 0 \quad \text{if} \quad m = \pm 1. \tag{63b}$$

Second, Suydam's condition cannot be fulfilled at $r = 0$, since the left-hand side of the inequality (33) necessarily vanishes there; and third, the minimization theorems of the last section are not applicable to an interval bounded on the left by $r = 0$. But these differences are not essential. As one can easily verify, the Euler–Lagrange solutions never oscillate at $r = 0$, so that Suydam's condition does not need to be fulfilled; and special minimization theorems can be proved for intervals $0 < r < r_2$ by the same methods that were used for intervals $r_1 < r < r_2$.[26] Without going into details, we merely give the result.

THEOREM 11. Necessary and sufficient conditions for the stability of a columnar pinch are given by Theorem 10 and its corollaries with the following modifications: (1) An Euler–Lagrange solution is said to be small at $r = 0$ if it satisfies the boundary conditions (63). (2) Although Suydam's condition must still be fulfilled for every $r > 0$, it need not be fulfilled at $r = 0$.

COROLLARY 11-1. A columnar pinch has the same stability criterion as a tubular pinch in the limit as $a \to 0$, $B_\theta(a) = 0$. Thus the stability of a columnar pinch is not changed if an infinitely thin wire with no current is placed along the axis.

PROOF. Although it is obvious that the wire can have no effect on stability for $m \neq \pm 1$, we might expect it to stabilize the $m = \pm 1$ displacements because of the added constraint that ξ must vanish at a conducting surface. That such a stabilizing effect does not in fact occur will be shown by examining the Euler–Lagrange solutions in the limit as $a \to 0$. For $m = \pm 1$ these solutions behave in the neighborhood of $r = 0$ like $1 + Ar^2$ and $1/r^2$, where A is some constant. For small but finite a, therefore, the solutions that vanish at $r = a$ are constant multiples of

$$\xi(r) = 1 + Ar^2 - \frac{a^2}{r^2}(1 + Aa^2). \tag{64}$$

[26] The cases $m = 0$, $m = \pm 1$, and $|m| > 1$ must be treated separately, since f and g depend on different powers of r in these three cases.

In the limit as $a \to 0$ this solution approaches $1 + Ar^2$, which satisfies Eq. (63b), so that the stability criterion of Theorem 10 approaches that of Theorem 11.

Corollary 11-2. A necessary condition for the stability of a columnar pinch is that the plasma pressure should have a minimum at $r = 0$ (*29*). (Because the ohmic heating is greatest at a finite distance from the axis, this condition is generally fulfilled in practice, even though the density has a maximum at $r = 0$.)

Proof. For small but nonvanishing values of r Suydam's condition reduces to $dP/dr > 0$, since this term is of order r while the other term is only of order r^3.

Theorems 10 and 11 solve the problem that was posed in the introduction. That is to say, they furnish a definite procedure for determining stability, since the question whether an Euler–Lagrange solution vanishes anywhere in the interior of I can always be answered by solving the Euler–Lagrange equation numerically. Furthermore, Theorem 1 enables us, if we are not interested in the stability of individual values of m and k but only in whether the pinch is stable for all values, to restrict our attention to the special cases $m = 0, k \to 0$ and $m = 1, -\infty < k < \infty$.

Another necessary and sufficient condition for stability is given by

Theorem 12. For specified values of m and k, a linear pinch is stable in an independent subinterval I if and only if: (1) Suydam's condition is fulfilled at the endpoints r_1 and r_2 if they are singular and if $r_1 \neq 0$. (2) If $\xi_1(r)$ and $\xi_2(r)$ are the Euler–Lagrange solutions satisfying

$$\xi_1(r) \text{ small at } r_1, \qquad \xi_1(r_0) = 1, \tag{65a}$$

$$\xi_2(r) \text{ small at } r_2, \qquad \xi_2(r_0) = 1, \tag{65b}$$

where r_0 is some interior point of I, and if $\xi_0(r)$ is defined by

$$\xi_0(r) = \begin{matrix} \xi_1(r), & r_1 < r < r_0 \\ \xi_2(r), & r_0 < r < r_2 \end{matrix}, \tag{66}$$

then $\xi_0(r)$ does not vanish anywhere in the interior of I. (3) The energy integral satisfies the inequality $W(r_1, r_2; \xi_0) > 0$, which is equivalent because of Eq. (26) to

$$\xi_1'(r_0) > \xi_2'(r_0). \tag{67}$$

Proof. It follows immediately from Theorem 9 and its corollaries that parts (1) and (2) are necessary, and the necessity of part (3) is obvious. To prove sufficiency, we note first of all that $W(r_1, r_2; \xi)$ is positive whenever $\xi(r_0) = 0$ because of Corollaries 4-1 and 6-1, so that our attention can be restricted to displacements for which $\xi(r_0) \neq 0$, and in particular to the ones that are normalized to unity at r_0. We shall therefore minimize $W(r_1, r_2; \xi)$ with respect to the class of displacements satisfying the normalization condition $\xi(r_0) = 1$, the boundary condition

$$\xi(r_1) = 0 \quad \text{if } r_1 = a, \tag{68a}$$

$$\xi(r_1) \text{ finite} \quad \text{if } r_1 \text{ is a singular point}, \tag{68b}$$

and a similar boundary condition at r_2. Since $\xi_0(r)$ does not vanish in the interior of I, it follows from Theorem 4 if $r_1 = a$ and from Theorem 6 if r_1 is singular that $W(r_1, r_0; \xi)$ is minimized by ξ_0. By applying Theorem 2 and its corollary we see that $W(r_0, r_2; \xi)$ is also minimized by ξ_0. The trial function ξ_0 therefore minimizes W over the entire interval I, and the pinch is stable if and only if $W(r_1, r_2; \xi_0) > 0$.

VI. THE SHARP PINCH

Let us now suppose that the plasma current is restricted to a layer of thickness δ and apply Theorem 12 in the limit as $\delta \to 0$ to obtain the necessary and sufficient condition for stability of a sharp pinch. We assume that the current layer extends from r_{0i} to r_{0e}, where $r_{0e} - r_{0i} = \delta$, and that the functions B_θ, B_z, and P are continuous at r_{0i} and r_{0e}. These functions are of order unity throughout the layer, but their derivatives are of order δ^{-1}. The regions $a < r < r_{0i}$ and $r_{0i} < r < b$ contain vacuum fields with components $B_{\theta i} r_{0i}/r$, B_{zi} and $B_{\theta e} r_{0e}/r$, B_{ze} respectively. Furthermore, the plasma pressure P is uniform in these regions. Equation (2) can be integrated across the layer in the limit as $\delta \to 0$, and we obtain the pressure balance equation

$$P_i + \tfrac{1}{2}B_i^2 = P_e + \tfrac{1}{2}B_e^2, \tag{69}$$

where $B^2 = B_\theta^2 + B_z^2$ and the subscripts i and e refer to r_{0i} and r_{0e} respectively.

Suydam's condition is given by the inequality (5), which is obviously satisfied throughout the constant pressure regions. It can also be written in the form

$$\frac{rB^2}{8\sin^2\phi}\left(\frac{d\phi}{dr}\right)^2 + \left[\frac{dP}{dr} - \frac{B^2}{4}\frac{\cos\phi}{\sin\phi}\frac{d\phi}{dr}\right] + \frac{B^2}{8r}\cos^2\phi > 0, \tag{70}$$

where $\phi = \tan^{-1}(B_\theta/B_z)$ is the angle between the magnetic field and the z axis. In the current layer the three terms of (70) are of order δ^{-2}, δ^{-1}, and 1 respectively, and the first term is necessarily positive; hence Suydam's condition is fulfilled everywhere unless $d\phi/dr$ vanishes at some point where P is a decreasing function of r.[27] Let us therefore assume that $d\phi/dr$ never vanishes in a region of

[27] We shall not worry about what happens when B^2 vanishes at some point r_1. Since B^2 is positive on each side of r_1, any pinch with $B^2(r_1) = 0$ can be regarded as the limiting form of a pinch in which B^2 never vanishes. For the same reason, we can assume that $d\phi/dr$ changes sign whenever it vanishes.

decreasing pressure. (The quantity $d\phi/dr$ measures the rate of rotation of the magnetic field as we pass through the layer, and the vanishing of $d\phi/dr$ means that the field reverses its sense of rotation.)

The next step in applying Theorem 12 is to find the Euler–Lagrange solutions $\xi_1(r)$ and $\xi_2(r)$ in each independent subinterval. Suppose first that there is at least one singular point in the layer but none in the constant-pressure interval $a < r < r_{0i}$, let r_s be the first singular point, and consider the independent subinterval $a < r < r_s$, taking $r_0 = r_{0i}$. In the constant-pressure part of this interval we can transform the Euler–Lagrange equation into Bessel's equation by a change of variable, obtaining the general solution

$$\xi(r) = \frac{kr}{krB_{zi} + \dfrac{mr_{0i}}{r}B_{\theta i}}[c_1I_m'(kr) + c_2K_m'(|kr|)], \tag{71}$$

where c_1 and c_2 are constants of integration. Our notation for Bessel functions is the same as Dwight's (*30*). Using the boundary condition $\xi_1(a) = 0$, we find that

$$\xi_1(r) = \frac{ckr}{krB_{zi} + \dfrac{mr_{0i}}{r}B_{\theta i}}[I_m'(kr)K_m'(|ka|) - K_m'(|kr|)I_m'(ka)], \tag{72}$$

where c is a normalization factor. Choosing c so that $\xi_1(r_{0i}) = 1$, we obtain

$$\xi_1'(r_{0i}) = \frac{1}{r_{0i}}L(r_{0i}, a), \tag{73}$$

where

$$\begin{aligned} L(r_1, r_2) &= \left.\frac{mB_\theta - krB_z}{mB_\theta + krB_z}\right|_{r=r_1} \\ &+ \frac{k^2r_1^2 + m^2}{k^2r_1^2}\,\frac{kr_1I_m(kr_1)K_m'(|kr_2|) - |kr_1|K_m(|kr_1|)I_m'(kr_2)}{I_m'(kr_1)K_m'(|kr_2|) - K_m'(|kr_1|)I_m'(kr_2)}. \end{aligned} \tag{74}$$

It can easily be shown fiom elementary properties of the Bessel functions that $\xi_1(r)$ never vanishes between a and r_{0i}.

To find $\xi_2(r)$, the solution in the current layer, let ϵ be some length such that $\delta^2/r_0 \ll \epsilon \ll \delta$, and start by considering the interval $r_s - \epsilon < r < r_s$. The approximations $f \cong \alpha(r - r_s)^2$ and $g \cong \beta$ are good throughout this interval, and since $\xi_2(r)$ is small at r_s, we obtain

$$\xi_2(r) \cong A\left(\frac{r_s - r}{\epsilon}\right)^{-n_1}, \tag{75}$$

where A is some constant of order unity and n_1 is the small root of Eq. (30). One can easily see from Eqs. (32) that α and β are of order δ^{-2} and δ^{-1} respectively. Equation (30) therefore reduces to $n_1 \cong -\beta/\alpha$, which is of order δ.

Let us next consider the interval $r_{0i} < r < r_s - \epsilon$, for which the right-hand boundary condition is that $\xi_2(r)$ and its derivative should both be continuous:

$$\xi_2(r_s - \epsilon) \cong A, \tag{76a}$$

$$\xi_2'(r_s - \epsilon) \cong An_1/\epsilon. \tag{76b}$$

We assert that the approximation $\xi_2(r) \cong A$ is valid throughout the interval now under consideration, and to verify this assertion we write

$$\frac{d}{dr}\left(f\frac{d\xi_2}{dr}\right) = g\xi_2 \cong Ag. \tag{77}$$

It follows immediately that

$$\frac{1}{A}f\frac{d\xi_2}{dr} \cong n_1\alpha\epsilon - \int_r^{r_s-\epsilon} g\,dr. \tag{78}$$

Now n_1, α, and g are of order δ, δ^{-2}, and δ^{-1} respectively. We can therefore neglect the term $n_1\alpha\epsilon$ and set the upper limit of integration equal to r_s with errors of order ϵ/δ,[28] which is small because the main term is of order unity. The result is

$$\frac{1}{A}f\frac{d\xi_2}{dr} \cong -\int_r^{r_s} g\,dr. \tag{79}$$

In the neighborhood of r_s we have $f \cong \alpha(r - r_s)^2 \sim (r - r_s)^2/\delta^2$; hence the integration of Eq. (79) leads to

$$\xi_2(r) - A \sim \int^{r_s-\epsilon} \frac{\delta^2\,dr}{(r - r_s)^2} \sim \frac{\delta^2}{\epsilon}. \tag{80}$$

Since δ^2/ϵ is small, the solution $\xi_2(r)$ is indeed close to A, and Eq. (79) is approximately valid everywhere between r_{0i} and $r_s - \epsilon$. Choosing $A = 1$ to satisfy the second part of Eq. (65b), we obtain

$$f(r_{0i})\xi_2'(r_{0i}) \cong -\int_{r_{0i}}^{r_s} g\,dr. \tag{81}$$

This equation is exact in the limit as ϵ/δ and δ^2/ϵ both approach zero.

It is clear that the trial function $\xi_0(r)$ given by Eq. (66) does not vanish

[28] As a matter of fact, these two errors cancel to lowest order, so that the remaining error is even smaller than ϵ/δ. This cancellation, however, is not needed in the proof.

anywhere between a and r_s. We can therefore substitute Eqs. (73) and (81) into the inequality (67) to obtain the stability criterion

$$\frac{1}{r_{0i}} f(r_{0i})L(r_{0i}, a) + \int_{r_{0i}}^{r_s} g\, dr > 0. \tag{82}$$

Now suppose that there is a singular point $r_s^{(0)}$ between a and r_{0i}. (There cannot be more than one because $k + m\mu$ is monotonic.) Since the Euler–Lagrange solution (72) never vanishes in the independent subinterval $a < r < r_s^{(0)}$, the pinch is stable in that interval according to Theorems 10 and 11. In the independent subinterval $r_s^{(0)} < r < r_s$, we must use the Euler–Lagrange solution that is small at $r_s^{(0)}$; the stability criterion is therefore given by (82) with $L(r_{0i}, r_s^{(0)})$ in the place of $L(r_{0i}, a)$. This means that the singularity at $r_s^{(0)}$ exerts the same stabilizing influence as would a conducting wall at the same point. Let us write

$$a_{\text{eff}} = \begin{cases} r_s, & \text{if } r_s \text{ is a singular point between } a \text{ and } r_{0i}, \\ a, & \text{if there is no singular point between } a \text{ and } r_{0i}. \end{cases} \tag{83}$$

The stability criterion is then given in both cases by (82) with a_{eff} in the place of a.

If r_s now designates the last singular point in the current layer, the stability criterion for the interval $r_s < r < b$ is completely analogous to (82). It is

$$-\frac{1}{r_{0e}} f(r_{0e})L(r_{0e}, b_{\text{eff}}) + \int_{r_s}^{r_{0e}} g\, dr > 0, \tag{84}$$

where b_{eff} is defined by analogy with a_{eff}. If there is more than one singular point in the layer, let $r_s^{(1)}$ and $r_s^{(2)}$ be two successive ones, and pick r_0 somewhere between $r_s^{(1)}$ and $r_s^{(2)}$. By an argument similar to the one leading to Eq. (81) we obtain

$$f(r_0)\xi_1'(r_0) \cong \int_{r_s^{(1)}}^{r_0} g\, dr, \tag{85a}$$

$$f(r_0)\xi_2'(r_0) \cong -\int_{r_0}^{r_s^{(2)}} g\, dr. \tag{85b}$$

As before, the trial function $\xi_0(r)$ never vanishes. The stability criterion is therefore given by the inequality (67), which now reduces to

$$\int_{r_s^{(1)}}^{r_s^{(2)}} g\, dr > 0. \tag{86}$$

Finally, suppose there are no singular points in the layer. We must then apply

Theorem 12 to the independent subinterval $a_{\text{eff}} < r < b_{\text{eff}}$, taking r_0 at some interior point of the layer. Using the fact that $\xi_1(r)$ and its derivative are continuous at the inner edge of the layer, and proceeding just as in the derivation of Eq. (81), we obtain

$$f(r_0)\xi_1'(r_0) \cong \frac{1}{r_{0i}} f(r_{0i})L(r_{0i}, a_{\text{eff}}) + \int_{r_{0i}}^{r_0} g\, dr. \tag{87a}$$

Similarly,

$$f(r_0)\xi_2'(r_0) \cong \frac{1}{r_{0e}} f(r_{0e})L(r_{0e}, b_{\text{eff}}) - \int_{r_0}^{r_{0e}} g\, dr. \tag{87b}$$

We again find that $\xi_0(r)$ never vanishes, and the stability criterion reduces to

$$\frac{1}{r_{0i}} f(r_{0i})L(r_{0i}, a_{\text{eff}}) - \frac{1}{r_{0e}} f(r_{0e})L(r_{0e}, b_{\text{eff}}) + \int_{r_{0i}}^{r_{0e}} g\, dr > 0. \tag{88}$$

The following expressions for f and g are valid to lowest order in δ throughout the layer:

$$f = \frac{r_0(kr_0B_z + mB_\theta)^2}{k^2r_0^2 + m^2}, \tag{89}$$

$$g = \frac{2k^2r_0^2}{k^2r_0^2 + m^2}\frac{dP}{dr}. \tag{90}$$

(We need not distinguish here between r_{0i} and r_{0e} because their difference is only of order δ.) The integrations in the inequalities (82), (84), (86), and (88) are therefore easily carried out. We obtain from (86), for example,

$$P(r_s^{(2)}) > P(r_s^{(1)}), \tag{91}$$

which leads to part (2) of the following theorem. Similarly, the other inequalities lead to parts (3) and (4).

THEOREM 13. A sharp linear pinch is stable if and only if: (1) The quantity $d\phi/dr$ does not vanish at any point in the current layer where dP/dr is negative. (2) If r_1 and r_2 are any two points in the current layer such that $\phi(r_2) - \phi(r_1)$ is an integral multiple of π, then $P(r_2) > P(r_1)$ if $r_2 > r_1$. (3) For all values of m and k such that there are no singular points in the current layer, we have

$$(kr_0B_{zi} + mB_{\theta i})^2L(r_{0i}, a_{\text{eff}}) - (kr_0B_{ze} + mB_{\theta e})^2L(r_{0e}, b_{\text{eff}}) + 2k^2r_0^2(P_e - P_i) > 0. \tag{92}$$

(4) For all values of m and k such that there is at least one singular point in the current layer, we have

$$(kr_0B_{zi} + mB_{\theta i})^2L(r_{0i}, a_{\text{eff}}) + 2k^2r_0^2[P(r_s^{(1)}) - P_i] > 0 \tag{93a}$$

and

$$-(kr_0B_{ze} + mB_{\theta e})^2L(r_{0e}, b_{eff}) + 2k^2r_0^2[P_e - P(r_s^{(2)})] > 0, \tag{93b}$$

where $r_s^{(1)}$ and $r_s^{(2)}$ are respectively the first and last singular points in the layer.

As was pointed out in the introduction, the stability criterion given by this theorem depends critically on the detailed structure of the current layer; it obviously cannot be obtained by simply treating the current-density components as delta functions.

Let us now suppose that the plasma pressure vanishes in one or both of the constant-pressure regions. What would be the effect of replacing the infinitely conductive pressureless plasma with a vacuum? The energy integral for a hydromagnetic system with a vacuum region is given by

$$W = W_f + W_s + W_v, \tag{94}$$

in which the three terms represent contributions from the plasma, the interface, and the vacuum respectively (*1*). The plasma contribution is given by Eq. (3) with the integration restricted to the region occupied by the plasma; it reduces as before to Eq. (14). The interface contribution is

$$W_s = \tfrac{1}{2} \int d\sigma(\mathbf{n}\cdot\boldsymbol{\xi})^2\mathbf{n}\cdot\left[\nabla\left(P + \tfrac{1}{2}B^2\right)\right], \tag{95}$$

where $d\sigma$ is the element of surface area, $\mathbf{n}$ is the unit normal pointing into the vacuum, and the bracketed expression is the jump in $\nabla(P + \frac{1}{2}B^2)$ as we pass through the interface into the vacuum. Using Eq. (2), we obtain

$$[(d/dr)(P + \tfrac{1}{2}B^2)] = -[B_\theta^2/r], \tag{96}$$

which vanishes because B_θ is continuous at the edges of the current layer; hence $W_s = 0$. The vacuum contribution is

$$W_v = \tfrac{1}{2} \int d^3x\, (\nabla \times \mathbf{A})^2, \tag{97}$$

where $\mathbf{A}$ is the vector potential of the perturbed vacuum magnetic field. We are to minimize with respect to $\mathbf{A}$ and $\boldsymbol{\xi}$ independently, except that the boundary condition

$$\mathbf{n} \times \mathbf{A} = -(\mathbf{n}\cdot\boldsymbol{\xi})\mathbf{B} \tag{98}$$

must be satisfied.

Let us extend the definition of $\boldsymbol{\xi}$ into the vacuum region by writing $\mathbf{A} = \boldsymbol{\xi} \times \mathbf{B}$.[29] Then W_v has the same form as W_f, and the boundary condition (98) reduces

[29] *Note added in proof:* This is made possible by imposing the gauge condition $\mathbf{A}\cdot\mathbf{B} = 0$.

to the continuity of ξ_r at the interface. The expression for the complete energy integral is therefore unchanged if the pressureless plasma is replaced by a vacuum. The only difference is that ξ, since it no longer represents the displacement of a real fluid, can be singular as long as the perturbed magnetic field $\mathbf{Q} = \nabla \times \mathbf{A} = \nabla \times (\xi \times \mathbf{B})$ is well-behaved. It follows that the stability criterion is the same for a vacuum as it is for a pressureless plasma if there are no singular points in the vacuum region. If there is a singular point, the Euler–Lagrange solution given by Eq. (72) (or by its analog for the outer region $r_0 < r < b$) becomes infinite aι that point. We have seen in the case of a pressureless plasma that this requires us to use a_{eff} instead of a in the stability criterion. With a vacuum, however, the singular behavior of ξ is immaterial[30]; what we must examine is the behavior of $\mathbf{Q}$. Because of Eqs. (6c), (10), and (13b), t is easy to express $\mathbf{Q}$ as a functional of ξ alone. Using the relations

$$(dB_z/dr) = (d/dr)(rB_\theta) = 0, \tag{99}$$

we obtain

$$Q_r = i(krB_z + mB_\theta)\xi/r, \tag{100a}$$

$$Q_\theta = \frac{-m}{k^2r^2 + m^2}\frac{d}{dr}[\xi(krB_z + mB_\theta)], \tag{100b}$$

$$Q_z = (kr/m)Q_\theta, \tag{100c}$$

which remains finite at the singular point. It follows that the singular point has no effect on the stability criterion for a vacuum, and we have

THEOREM 14. The stability criterion for a sharp linear pinch with a vacuum in the region $a < r < r_0$ (or $r_0 < r < b$) is given by Theorem 13 with $P_i = 0$ (or $P_e = 0$) and a in the place of a_{eff} (or b in the place of b_{eff}).

Thus a vacuum and a pressureless plasma are equally stable if there is no singular point in the pressureless region, but the pressureless plasma is more stable if there is such a point. This result is consistent with the comparison theorem in Section 3b of Ref. *1*. Also, the stability criterion given by Theorem 14 agrees with that of Rosenbluth (*21*) for the sharp columnar pinch if P_e, $B_{\theta i}$, and a are set equal to zero and with that of Newcomb and Kaufman (*22*) for the sharp tubular pinch if P_i and P_e are set equal to zero. We note that the results of this section provide a rigorous justification for the heuristic method used by Newcomb and Kaufman, which was based on the assumption that the minimizing displacement $\xi_0(r)$ is constant in the current layer. It has been shown that $\xi_0(r)$ is indeed constant except in the small intervals $r_s - \epsilon < r_s < r_s + \epsilon$, and one can easily verify that these intervals do not contribute appreciably to

[30] It does, however, prevent us from integrating Eq. (14) by parts to obtain Eq. (15); the simple form of the energy integral given by Eq. (15) is therefore invalid for a vacuum region containing a singular point.

$W(\xi)$. Thus $W(\xi)$ is not changed in lowest order if $\xi_0(r)$ is replaced by a constant.

We have obtained the stability criterion for a sharp pinch from Theorem 12, which involves two Euler–Lagrange solutions, ξ_1 and ξ_2, in each independent subinterval. It may also be instructive to show how it can be obtained directly from Theorem 10, which involves only one Euler–Lagrange solution.[31] Assuming, for example, that there is at least one singular point in the current layer but none between a and r_{0i}, let us consider the independent subinterval $a < r < r_s$, where r_s is the first singular point. The relevant Euler–Lagrange solution is given by Eq. (72) between a and r_{0i}. Requiring $\xi(r)$ and its derivative to be continuous at r_{0i}, we can find the solution in the interval $r_{0i} < r < r_s - \epsilon$ by an argument similar to the one leading to Eq. (79). The result is

$$\xi \cong 1, \tag{101a}$$

$$f\frac{d\xi}{dr} \cong \frac{1}{r_{0i}} f(r_{0i})L(r_{0i}, a) + \int_{r_{0i}}^{r} g\, dr. \tag{101b}$$

In the interval $r_s - \epsilon < r < r_s$ we have $f \cong \alpha(r_s - r)^2$, $g \cong \beta$, and for the general solution of the Euler–Lagrange equation:

$$\xi(r) \cong c_1(r_s - r)^{-n_1} + c_2(r_s - r)^{-n_2}, \tag{102}$$

where n_1 and n_2 are determined by Eq. (30). The relevant orders of magnitude are $\alpha \sim \delta^{-2}$, $\beta \sim \delta^{-1}$, and $n_1 = 1 - n_2 \sim \delta$. The constants c_1 and c_2 are determined by the boundary conditions

$$\xi(r_s - \epsilon) \cong 1, \tag{103a}$$

$$\xi'(r_s - \epsilon) \cong A/\alpha\epsilon^2, \tag{103b}$$

where

$$A = \frac{1}{r_{0i}} f(r_{0i})L(r_{0i}, a) + \int_{r_{0i}}^{r_s} g\, dr. \tag{104}$$

Since the second term of Eq. (102) dominates when r is sufficiently close to r_s, the solution $\xi(r)$ goes through zero if and only if c_2 is negative. Substituting (102) into the boundary conditions (103), and solving for c_2 to lowest order in ϵ/δ and δ^2/ϵ, we obtain $c_2 \cong A/\alpha$. Since α is necessarily positive, the stability criterion for the independent subinterval $a < r < r_s$ is that A should be positive, which is the same as our previous result, the inequality (82).

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[31] This will be equivalent in its essentials to Rosenbluth's treatment of the sharp columnar pinch (*21*).

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Stability and Heating in the Pinch Effect

By M. N. Rosenbluth *

One of the most promising types of thermonuclear device is the stabilized pinch.[1, 2] This consists of a pinched cylindrical plasma in which a longitudinal magnetic field is trapped. This field provides rigidity against the various types of instability to which the pinch is subject. An external conductor which confines the magnetic field of the pinch provides additional stability, so that, with a proper choice of parameters to define the equilibrium, the configuration may be made linearly stable against all perturbations.

In the first part of this paper we shall discuss some surface instabilities which may arise in the stabilized pinch. In the second part we shall discuss the disassembly and heating of the plasma which results from collisions.

SURFACE-LAYER INSTABILITIES

Previous calculations of stability[1, 2] have dealt mainly with an equilibrium with a very sharp surface layer such as might be expected with a very highly conducting plasma.[3] That is, for $0 < r < r_0$, we have

$B_\theta = 0$; $B_Z = \alpha_p B_0$; $p = (1 + \alpha_v^2 - \alpha_p^2)(B_0^2/8\pi)$

and for $\beta r_0 > r > r_0 + \delta$, we have

$B_\theta = B_0 r_0/r$; $B_Z = \alpha_v B_0$; $p = 0$.

(There is a glossary of symbols at the end of the paper.)

The radius of the external conductor is βr_0. In the thin region between r_0 and $r_0 + \delta$, large surface currents must flow. However, it would appear at first sight that in the limit of small δ this complicated region need not be considered explicitly, since, in the dynamic equations for the perturbation, one may use the conditions of continuity of the normal components of the magnetic field and stress tensor to relate the interior solutions to the exterior ones. We shall now try to give a more complete discussion of the above-defined problem.

It has been shown previously[4] that if the magnitude of B does not change along a field line, as is the case here, and if all distances are large compared with a Larmor radius and we confine ourselves to the marginal-stability case $\omega = 0$, then the standard magnetohydrodynamic equations are valid. That is, the governing equations for the perturbation are

$$\nabla \delta p = [(\nabla \times \mathbf{B}) \times \delta\mathbf{B} + (\nabla \times \delta\mathbf{B}) \times \mathbf{B}]/4\pi \qquad (1)$$

$$\nabla \cdot \delta\mathbf{B} = 0, \qquad (2)$$

where $\delta\mathbf{B}$ and δp are the perturbed quantities and $\mathbf{B}$ is the equilibrium magnetic field.

The boundary conditions are regularity at the origin and $\delta B_r = 0$ at $r = \beta r_0$. For the simple cylindrical geometry of the problem, the perturbations may be expanded in normal modes, i.e.,

$$\delta p,\ \delta\mathbf{B} = (p_1, \mathbf{B}_1) e^{i(kZ + m\theta)}. \qquad (3)$$

With this substitution, Eqs. (1) and (2) may be easily reduced to a single equation as follows:

Let

$$X = rB_{1r}/[kB_Z + (m/r)B_\theta],$$

then

$$\frac{d}{dr} F(r) \frac{dX}{dr} + G(r)X = 0, \qquad (4)$$

where

$$F(r) = \frac{r}{m^2 + k^2 r^2}\left(kB_Z + \frac{m}{r} B_\theta\right)^2 \qquad (5a)$$

and

$$G(r) = \frac{2B_\theta}{r^2}\left(\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right) - \frac{1}{r}\left(kB_Z + \frac{m}{r} B_\theta\right)^2 - \frac{(2mB_\theta)^2}{r^3(m^2 + k^2 r^2)} - \frac{d}{dr}\left\{\frac{2mB_\theta[kB_Z + (m/r)B_\theta]}{r(m^2 + k^2 r^2)}\right\}. \qquad (5b)$$

Since we are considering the marginal-stability problem, it follows that if the solution of Eq. (4) which is regular at the origin should vanish at the external conductor, then the equilibrium is neutrally stable to the perturbation being considered. If the solution crosses the axis before reaching the external conductor, this implies instability, since we should have to move the conductor in to stabilize the system.

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For the general equilibrium specified by arbitrary $B_\theta(r)$ and $B_Z(r)$, Eq. (4) must, of course, be numerically integrated, although useful limits may be obtained from the variational expression [5, 6] corresponding to Eq. (4). A code for the numerical investigation of Eq. (4) has been prepared. However, for the sharp-layer case, i.e., $\delta \ll r_0$, we may obtain analytic results using the following procedure:

For $r < r_0$ and $r > r_0 + \delta$, i.e., where there is no current, Eq. (4) may be solved explicitly in terms of Bessel functions. This gives us values for the logarithmic derivatives which must be joined on to the solution in the surface layer.

Within the surface layer we can neglect $1/r$ compared to $\partial/\partial r$ and rewrite Eq. (4) in the form

$$\frac{d}{dr}(u + H) = \frac{u^2}{r_0^2 F}, \tag{6}$$

where

$$u = \frac{r_0^2 F}{B_0^2 X}\frac{dX}{dr}$$

and

$$H = \frac{(k^2 r_0^2 - m^2)B_\theta^2 - 2mkrB_Z B_\theta}{(m^2 + k^2 r^2)B_0^2}. \tag{7}$$

In the region of interest, B_θ and B_Z are changing rapidly. It is apparent from Eq. (5) that for some ranges of k and m, depending on α_v, $F(r)$ will pass through zero, so that Eq. (6) develops a singularity. Parenthetically we may remark that because of this singularity there is some difficulty about the derivation of the magnetohydrodynamic Eq. (1). This question is receiving further study, but for the purpose of this paper we regard the conventional mode as valid.

For modes which have no singularity, the right-hand side is negligible over the small distance δ, and we find as the stability criterion

$$[u + H]_{r_0} > [u + H]_{r_0+\delta}\,. \tag{8}$$

For reference purposes, the explicit forms of these functions are

$$[u + H]_{r_0} = \alpha_p^2 (kr_0)^2 \frac{I_m(kr_0)}{(kr_0)I_m'(kr_0)}$$

$$[u + H]_{r_0+\delta} = (m + \alpha_v kr_0)^2 \times \frac{G_{\beta,m}(kr_0)\dfrac{I_m(kr_0)}{(kr_0)I_m'(kr_0)} - \dfrac{K_m(kr_0)}{(kr_0)K_m'(kr_0)}}{1 - G_{\beta,m}(kr_0)}$$

where

$$G_{\beta,m}(kr_0) = \frac{K_m'(\beta kr_0)}{K_m'(kr_0)}\,\frac{I_m'(kr_0)}{I_m'(\beta kr_0)}\,. \tag{9}$$

Equation (8) is, of course, identical with the old stability calculation.

In discussing the singular case it is convenient to consider the situation as we integrate towards the singularity. Until we get very close to the singularity—more specifically, until $F \sim \delta$—the quantity $u + H$ will remain constant, as before. This determines the value of u as we approach the singularity. Very near the singularity we may neglect the term H', so that in this region

$$\frac{1}{u} + \int \frac{dr}{F} = \text{constant}.$$

Hence, $1/u$ becomes infinite at the singularity as the integral diverses. Moreover, depending on the sign of u as it approaches the singularity, $1/u$ may pass through zero. This corresponds to X crossing the axis, i.e., to instability. In order to avoid instability in this case we must then require that

$$[u + H]_{r_0} - [H]_s > 0, \tag{10a}$$

$$[H]_s - [u + H]_{r_0+\delta} > 0. \tag{10b}$$

Here, $[H]_s$ means the value of H at the singularity, which can easily be shown to be

$$[H]_s = [B_\theta^2]_s / B_0^2. \tag{11}$$

It may seem strange that two criteria must be satisfied for the stability of a single mode. The explanation is that the singularity in Eq. (6) is so strong that it completely separates the region interior to the singularity from the exterior. Thus, if we violate Eq. (10b), for example, the instability is essentially confined to the outside of the surface layer. The new stability criteria are considerably more complicated than the old one, since they depend on the structure of the surface layer. This structure may be described by giving B_θ^2 as a function of $\varphi = \tan^{-1} B_\theta/B_Z$, the pitch of the field. Equation (10) then gives upper and lower limits for B_θ^2. It will be noted that the value of φ at the singularity is simply related to k and m by $kr/m + \tan\varphi_s = 0$. The range of possible layer structures is further limited by the requirement that the plasma pressure be positive throughout. This gives

$$B_\theta^2 = B^2 \sin^2\varphi < B_0^2(1 + \alpha_v^2)\sin^2\varphi\,. \tag{12}$$

These results are plotted in Fig. 1 for $\alpha_v = \pm 0.25$. As usual, we need consider only $m = 0, \pm 1$.

Curve I is the function $u + H$ divided by α_p^2 evaluated at r_0. Curve II is $u + H$ evaluated at $r_0 + \delta$ with various β for $m = \pm 1$. The x's are the $m = 0$ values of $u + H$ evaluated at $r_0 + \delta$, evaluated at $k = 0$, the most dangerous case. Equation (8), which must be satisfied for all k and m, hence all φ, and for both values of α_v, requires that Curve II lie above Curve I. This means that $\beta = 2.5$ is unstable.

The more stringent conditions (10) and (12) need only be applied to values of k and m so that a singularity develops within the layer. This means, for the case of positive α_v, the region to the right of $\bar{\varphi} = \tan^{-1} 1/|\alpha_v|$ and, for negative α_v, to the left of this point. Condition (10) requires that at

$$\bar{\varphi} = \tan^{-1}[-(\alpha_v/|\alpha_v|)(B_\theta/B_z)]\,.$$

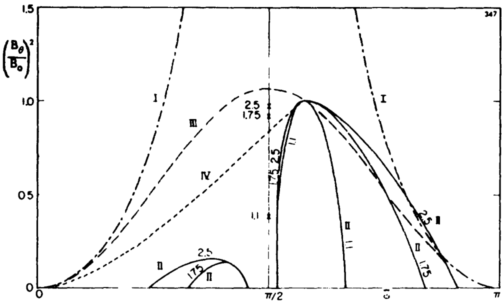

Figure 1. Stability diagram for $\alpha_v = \pm 0.25$ and $\beta = 1.1, 1.75, 2.5$

The shape of a thin surface layer is given by a graph of B^2_θ/B_0^2 vs $\overline{\varphi}$. To insure stability, this shape must be between certain limits, shown in this figure for the case $\alpha_v = |25|$, $\beta = 1.1, 1.75, 2.5$. Previous [1,2] stability criteria require only that curve I multiplied by α_p^2 must lie above curve II for all $\overline{\varphi}$ between O and π. In addition we must now require for the range of $\overline{\varphi}$ which occurs within the layer that the curve which defines the layer shape must itself lie between curve I (multiplied by α_p^2) and curve II and above the cross. All physical layers (with positive plasma pressure) must lie below curve III. It can be shown that curves III and II cross for all positive α_v so that no stable layer can be constructed

B_θ^2 must lie between Curves I and II and above the x. In addition, Eq. (12) requires B_θ^2 to be below Curve III. Hence, we deduce that there is no stable layer structure for $\alpha_v = +0.25$. Curve IV shows the type of layer which develops because of diffusion, as discussed below. This type of structure for $\alpha_v = -0.25$ is seen to be unstable against long-wavelength $m = 0$ modes, for the cases $\beta = 2.5$ and 1.75, and against $m = 1$ modes with pitch close to that of the external field.

It is, in fact, apparent that for positive α_v, i.e., the external field in the same direction as the internal field, no stable layers exist, since (10*b*) and (12) are incompatible. It appears that in most cases where Eq. (8) is satisfied and the field is reversed, some stable layers can exist. Moreover, their structure is not too improbable although it appears that the diffusion layer is likely to be unstable against long-wavelength $m = 0$ disturbances and $m = 1$ disturbances near the pitch of the external field. It will be noted that condition (10*b*) appears much more dangerous than (10*a*), so that the instabilities are limited to the outside regions of the plasma.

The importance of these instabilities is questionable. The bulk of the plasma is unaffected; moreover they are slow—the multiplication time is proportional to the surface layer width, and they cannot remain in the linear phase of growth for long. It appears likely that the surface will become unstable and then restabilize itself in a new equilibrium, perhaps helical rather than cylindrical. It is noteworthy that such helical equilibria tend to generate a reversed B_Z, as required for stability. Eventually, as we have noted, diffusion will make the layer unstable and the cycle must be repeated.

While we have treated only the sharp-layer case, it is clear that the same qualitative features persist for thick layers. The most important practical question would seem to be under what conditions the instability can proceed far enough to drive a significant amount of plasma into the walls, thereby releasing impurities. Experimental work with a reversed B_Z is in progress.

DISASSEMBLY AND HEATING

It is apparent that the stabilized pinch is not a true equilibrium when interparticle collisions are considered. These collisions imply a finite conductivity of the plasma,[7] so that electric fields must be present to drive the currents. These fields imply a change of flux, i.e., a relaxation of the crossed fields. At the same time, ohmic heating must occur, and temperature gradients will lead to kinetic thermal conduction. As a consequence of these involved processes, the plasma is simultaneously disassembled and heated. We shall find that a proper choice of parameters leads to favorable conditions for a thermonuclear reaction with no external heating mechanism. Qualitatively, this is because the crossed-field configuration represents a higher energy state of the magnetic field than the state which will prevail after diffusion has occurred and the components of the fields have been mixed. The excess energy goes into heating the plasma. Of course, a quantitative analysis is required to estimate the magnitude of the effect and to compare the disassembly and heating rates.

We therefore study the evolution of the equilibrium situation described at the beginning of this paper. A detailed solution will be presented for the case with the initial conditions $\delta = 0$, a sharp layer; $\alpha_p \approx 1$, a low-temperature plasma; and $\alpha_v = 0$, a 90° angle between the fields. It is apparent from the diffusion character of the problem that early departure from the initial values is confined to the neighborhood of the surface. For analytic simplicity we shall therefore consider a plane problem, calling the initial surface-layer position $X = 0$ and letting the plasma extend extend $-\infty$ and the vacuum field to $+\infty$.

The equations of motion [8] are: Maxwell's equations; the conservation of mass, momentum, and energy for the plasma; Ohm's Law; and the equation governing heat transport in the plasma.

These are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{13}$$

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j} \tag{14}$$

$$\frac{\delta \varrho}{\delta t} + \nabla \cdot \varrho \mathbf{v} = 0 \tag{15}$$

$$2\nabla(\varrho k T) = \mathbf{j} \times \mathbf{B} \tag{16}$$

$$\frac{\partial}{\partial t}\,(3\varrho kT + \boldsymbol{\nabla}\cdot(\mathbf{Q} + 5\varrho kT\mathbf{v}) - \mathbf{E}\cdot\mathbf{j} = 0 \qquad (17)$$

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} = \frac{\mathbf{j}}{\sigma} - \frac{(\mathbf{j}\cdot\mathbf{B})\,\mathbf{B}}{2B^2\sigma} - \frac{3}{2}\,\frac{\varrho k}{B^2\sigma}\,(\mathbf{B}\times\boldsymbol{\nabla}T) \qquad (18)$$

$$\mathbf{Q} = -\frac{\varrho^2 T}{B^2\sigma}\left\{\left(2\,\frac{m_i}{m_e}\right)^{\frac{1}{2}} k\boldsymbol{\nabla}T - \frac{3}{2\varrho}\,(\mathbf{j}\times\mathbf{B})\right\} \qquad (19)$$

$$\sigma = \frac{3}{8}\sqrt{\frac{2}{\pi}}\,\frac{(kT)^{\frac{3}{2}}}{e^2(m_e^{\frac{1}{2}})\ln\Lambda} = \sigma\,(kT)^{\frac{3}{2}} \qquad (20)$$

Here, **E** is the electric field, **B** the magnetic field, **j** the current density, ϱ the number density of ions or electrons, k the Boltzmann constant, v the plasma velocity, m_i and m_e the ion and electron masses, and Λ the ratio of maximum to minimum impact parameter.

In Eq. (16) we have used the fact that the diffusion velocities are very subsonic, so that hydrostatic equilibrium is maintained. (Needless to say, we are not considering possible unstable motions.) It is assumed throughout that the distribution functions are close to Maxwellian, with equal temperatures for electrons and ions. This is easily shown to be the case after the diffusion wave has penetrated a distance large compared to an ion Larmor radius. Equations (18) through (20) are, of course, valid only within this limit. They have been derived [8] from the transport equation by considering the small deviations from the Maxwell distribution which are necessary to compensate for the collision terms in the presence of field, density and temperature gradients. A convenient mathematical tool for this purpose is the Fokker-Planck Equation.[9]

For a plane problem with an initially sharp layer, the system of equation has a similarity solution of the variable $X/\sqrt{t}$. The resulting ordinary differential equations have been integrated on the General Atomic IBM-650 by Miss G. Roy. The solution is shown in terms of the appropriate dimensionless variables in Fig. 2.

The abscissa is the dimensionless similarity variable

$$\xi = \tfrac{1}{4}[\sigma B_0^3/(\pi\varrho_0^3)^{\frac{1}{2}}]^{\frac{1}{2}} X/t^{\frac{1}{2}}.$$

Here, X is the Lagrangian coordinate representing the initial position of the mass point, B_0 and ϱ_0 are the field strength and particle density in the undisturbed regions, φ is the angle the field makes with the original direction of the field in the plasma, and β is the dimensionless pressure, $\beta = 16\pi\varrho kT/B_0^2$.

It will be seen that the plasma pressure reaches a maximum value of about 0.43 near $\xi = 1$. The field is about half uncrossed at this point. For $\xi < 1$ the solution is nearly isothermal; for $\xi > 1$ the density remains almost constant. In the following discussion we will refer to $\xi = 1$ as the depth of penetration of

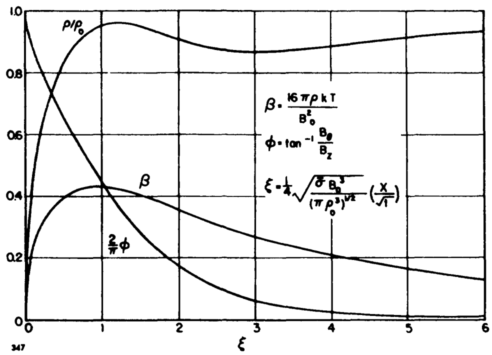

Figure 2. Solution of diffusion Equations 13-20 for the case $\alpha_p = 1$, $\alpha_v = 0$

The results of diffusion resulting from a case in which the plasma pressure is initially low compared with the magnetic pressure and in which the magnetic field in the vacuum is at right angles to the field in the plasma. Plasma pressure, plasma density, and magnetic field direction are plotted vs $X/\sqrt{t}$ where X is the Lagrangian distance from the original interface

the wave, and denote quantities at this point by the subscript $_p$.

The following features of the solution seem of some significance.

1. The stabilized pinch needs only to be preheated to a temperature consistent with the recombination rate (about 100 ev). From this point on, the heating occurs naturally through intermixing of the fields. The plasma comes to a temperature

$$T_p \approx .5B_0^2/16\pi\varrho_0. \qquad (21)$$

As mentioned earlier, this heating does not occur for uncrossed fields.[8]

2. The plasma–vacuum interface moves a distance of about 1.2 X_p into the space initially occupied by the vacuum field. Hence, in a typical pinch configuration no plasma will touch the wall until the wave has penetrated quite deeply.

3. The front of the wave is characterized by a nearly constant shear. As discussed in the first part of this paper, the resulting layer shape is linearly unstable against only a rather narrow region of wavelength perturbations if a negative B_z is provided.

4. The rate of penetration depends on the conductivity at the temperature of the heated plasma. In cgs units, with the temperature in kilovolts,

$$X_p = 18t^{\frac{1}{2}}T_p^{-\frac{3}{4}}. \qquad (22)$$

5. Similarly, the energy delivered to the plasma is

$$\frac{dE}{dt} = 25\,\frac{B_0^2}{8\pi}\,\frac{1}{t^{\frac{1}{2}}}\,T_p^{-\frac{3}{4}}\text{ergs/cm}^2\text{sec}. \qquad (23)$$

The other important mechanisms in the energy balance of the burning pinch are fusion and radiative loss; i.e., bremsstrahlung. Both of these produce energy according to

$$\frac{dE}{dt} = \varrho_0^2 F(T) \text{erg/cm}^3\text{sec.} \quad (24)$$

Thus, if the temperature is greater than a critical value, about 6 kev for D–T and 50 kev for D–D, the reaction is self-sustaining.[10] It may be well to operate only slightly above this critical value to that disassembly is not substantially faster than that given by Eq. (24). At a fixed temperature, the disassembly time is proportional to r_0^2, as may be seen from Eq. (22). Moreover, using Eqs. (24) and (21) we see that the burning time is proportional to B_0^{-2}. Hence, we should expect the efficiency to be a function of $B_0 r_0$ only, i.e., the current. In fact, it can be easily shown that if the quantities B_0^2/ϱ_0 and $B_0 r_0$ are held fixed, the complete set of equations for disassembly, heating, and burning can be simply scaled to a change in r_0. We may note that the currents must be at least 2×10^6 amp to contain the α particle resulting from the D–T reaction.

Finally, we indicate in Table 1 a possible D–T reactor design, taking an optimistic criterion for disassembly, i.e., $X_p = r_0$. The figures should be taken as only a rough indication.

Table 1. Possible Characteristics of a Diffusion-limited, Self-heated D–T Reactor

Quantity	*Value*	*Scaling*
Major radius of torus	30 cm (arbitrary)	r_0
Minor radius of torus	6 cm	r_0
Initial plasma radius (r_0)	1.5 cm	r_0
Current	3×10^6 amp	$(r_0)^0$
Pressure at wall	400 atm	r_0^{-2}
Initial pinched density (ϱ_0)	1.3×10^{17}/cm^3	r_0^{-2}
Burning temperature (T_p)	6 kev	$(r_0)^0$
Disassembly time	0.15 sec	r_0^2
Total magnetic energy	4×10^6 joule	r_0
Losses (copper torus)	2.5×10^6 joule	r_0
Energy produced	3×10^7 joule	r_0
Temperature rise of copper surface due to radiation	500°C	r_0^{-2}
Efficiency of burning	10%	$(r_0)^0$

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The qualitative features of the intermixing heating were realized independently by S. A. Colgate and J. L. Tuck. It is a particular pleasure to acknowledge some stimulating discussions with Dr. Colgate and with A. N. Kaufman, who independently had studied some of these questions.[8]

GLOSSARY OF SYMBOLS

r_0	radius of pinch
B_θ	unperturbed azimuthal field
B_Z	unperturbed longitudinal field
B_0	value of azimuthal field, B_θ, at r_0
α_p	ratio of internal longitudinal field, B_Z, to B_0.
α_v	ratio of external longitudinal field, B_Z, to B_0. α_v is defined as positive if the external field has the same sign as the internal one
β	(as used in first part) ratio of external conductor radius to r_0
p	plasma pressure
δ	thickness of surface layer
k	longitudinal wave number of perturbation
m	azimuthal wave number of perturbation
$\mathbf{B}_1$	perturbed magnetic field
X, F, G	defined in Eq. (4)
u, H	defined in Eq. (7)
$I(X), K(X)$	Bessel functions in Watson's notation
$[B_\theta^2]_s$	value of B_θ^2 at the radius where $F(r) = 0$
φ	$\tan^{-1}(B_\theta/B_Z)$
σ	plasma conductivity
β	(as used in second part) $16\pi\varrho kT/B_0^2$
ξ	a dimensionless variable defining the depth of penetration
T_p	temperature at $\xi = 1$
ϱ_0	undisturbed plasma density

Mr. Rosenbluth presented a survey, at the Conference, of Papers P/347 (above), P/354, P/1861, P/376 and P/2433:

Linear stability theory in a magnetohydrodynamic, collision-dominated fluid is a fairly well understood subject.[11] However, in a high-temperature plasma in which the particles interact only through the macroscopic fields, the situation is not so clear. Let us begin by discussing the types of waves characteristic of an infinite homogeneous plasma with constant magnetic field.

The situation is shown in Table 2. The magnetic field is taken to be in the Z direction, and we consider a wave propagating in the X, Z plane. The distribution function is an arbitrary function of the magnitude of velocity and its component parallel to the field. In general, four types of wave appear possible. One of these is a trivial mass flow in the direction of the field which causes no charge or

Table 2. Types of Plasma Waves

$B = B_z$; $\mathbf{E} = \mathbf{E} \exp\,[i\,(\omega t + k_X X + k_Z Z)]$; $f = f\,(v^2, v_Z)$

Type	**E**	*Character*	*Stability*
Plasma oscillation . .	$\mathbf{E}_k$	Electrostatic $\omega \approx \omega_p = \sqrt{(4\pi n e^2/m_e)}$ Landau damped	Overstability occurs if groups of electrons have a v_Z substantially exceeding mean thermal velocity.
Alfvén waves	$\mathbf{E}_x$	Transverse $\dfrac{\omega}{k_z} \approx \sqrt{\left(\dfrac{B^2}{4\pi\varrho}\right)}$ Undamped-Incompressible	Overstable for long wave-lengths if ion distribution is anistropic.
Hydromagnetic waves	$\mathbf{E}_y$	$\dfrac{\omega}{kX} \approx \sqrt{\left(\dfrac{B^2}{4\pi\varrho} + \mathrm{P}\right)}$	Unstable if $\dfrac{P_\perp^2}{P_\parallel} > \dfrac{B^2}{8\pi}$ or $P_\parallel - P_\perp > \dfrac{B^2}{4\pi}$

current. The other three are indicated in Fig. 3 with a qualitative discussion of their properties. The comments are not inclusive or exact.

If the distribution function is isotropic and a decreasing function of energy, it is easy to show on general statistical grounds that the plasma must be stable.[12] However, it appears that even in the infinite homogeneous case, small deviations in the distribution function may lead to alarming instabilities.

For the plasma oscillations, overstability, i.e., growing oscillations, occurs for a wavelength such that $k_Z U = \omega_p$, where U is the velocity of a non-thermal group of electrons, or of electrons and ions relative to each other.[13] Thus, these particles move in phase with the disturbance. It will be noted that the resonant particles see a non-oscillatory electric field and hence can move across magnetic field lines.

In the event that plasma heating is produced by electric fields parallel to the magnetic-field lines, we may expect that groups of high-energy electrons will be readily created because of the fall-off of cross section with energy. Thus, such a parallel field leads naturally to an unstable situation. The Alfvén-wave instability depends on a similar resonance for $\omega + k_Z v_Z =$ Larmor frequency.[13] It is particularly noteworthy since it occurs even for very small pressure anisotropies.

The hydromagnetic waves, which are the most nearly analogous to ordinary sound waves, become violently unstable for large pressure anisotropies.[14] In particular, a shock perpendicular to the field lines creates a large transverse pressure, $P_\perp$. The resulting

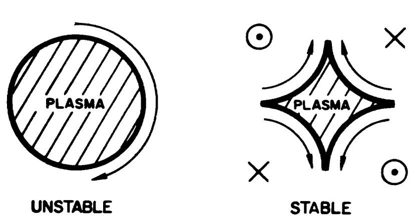

Figure 3

instability may play an essential role in producing the entropy necessary for the existence of the shock. Thus we see that there are many unstable situations even for a simple infinite plasma.

When we consider a finite geometry, the situation is, of course, much more complicated. Progress to date has been made largely by assuming that the characteristic scale of inhomogeneities is large compared to Larmor radius and Debye length and that frequencies are small compared to orbital and plasma frequencies. As we have heard in the paper presented by M. D. Kruskal,[15] a variational expression for stability with these approximations may be obtained which does not differ substantially from the fluid case. It should be noted that the plasma oscillation and Alfvén instabilities which we have discussed do not appear in this approximation. Thus, there has been no complete theoretical demonstration that any finite confined plasma can be stabilized.

However, even in this magnetohydrodynamic approximation it is difficult, although possible, to attain a stable static equilibrium. Perhaps the simplest situation to be studied is one in which the plasma contains no internal magnetic field. This constant-pressure plasma is then confined by an external magnetic field which must obey the condition that $B^2/8\pi =$ pressure along the surface. It has been shown that a necessary and sufficient condition for stability is that the principal normal to the surface must at all points be directed into the plasma.[16] This result is not restricted to the linear theory. The geometry is illustrated in Fig. 3. The shaded region in the diagram represents the plasma. On the left we have a convex unstable surface. This is the pinch. The cusp device on the right is concave at all parts and therefore stable. It appears to be the only possible confined stable equilibrium in which there is no field embedded in the plasma. Unfortunately, there is a finite rate of plasma loss through the cusps.

Another situation which has received a preliminary study is that of a plasma supported against gravity by a rotating magnetic field. The rotation slows down the instability but does not eliminate it.[16]

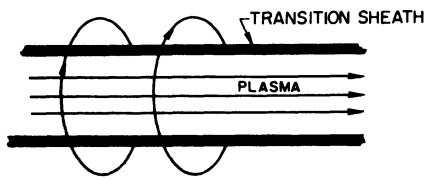

Figure 4. Stabilized pinch geometry

It is perhaps fair to say that the geometry which has received the most attention throughout the world is the stabilized pinch. It was realized early that the pinch was subject to the above-mentioned instability.[17] In order to correct the situation, an axial magnetic field was introduced into the plasma. This situation is shown in Fig. 4. It was then shown by several authors that complete stability could be obtained by proper choice of the internal field and the position of an external conducting shell.[18] However, these treatments neglected the structure of the current-carrying surface layer in which the field changes its direction. In practice, this sheath may be thick.

The equations of motion governing a perturbation of the form $e^{i(k_z+m\theta)}$ may be easily written. These equations develop a strong singularity at the radius where $kB_Z + (m/r)B_\theta = 0$, that is to say, at such a radius that the pitch of the perturbation matches the spiral of the unperturbed magnetic field. This is perhaps not surprising since at this radius the perturbation will not bend the field lines appreciably and the plasma can flow freely along the lines.

The mathematical effect of the singularity is that the region exterior to the singular point is completely separated from the interior of the plasma. This brings into existence a class of surface instabilities which are not affected by the internal stabilizing field. The essential character of the instability is an azimuthal bunching of the parallel current filaments which exist at a given radius.

A useful necessary condition for stability [19] is that at all points

$$\frac{B_Z^2}{8\pi}\frac{r}{4}\left[\frac{d}{dr}\ln\frac{B_\theta}{rB_Z}\right]^2 + \frac{dp}{dr} \geq 0.$$

Necessary and sufficient conditions have been found for the favorable case of a very thin transition layer.[20] It can be shown that only very special shapes of surface layer are stable. In particular, no stable surface can be found unless there is a longitudinal field outside the plasma which is opposite in direction to the internal field.

Let us now consider the diffusion and intermixing of the crossed magnetic fields due to interparticle collisions—without regard to stability. From the Rutherford cross section one can compute the various transport coefficients of the plasma—electrical and thermal conductivities and thermoelectric coefficients. Then the usual conservation equations—mass, momentum and energy—plus Maxwell's equations provide a complete set of dynamical equations for the diffusion process.[20] In particular, we study the case of an initially sharp surface layer which separates a region containing low-pressure plasma and axial magnetic field from a vacuum region with azimuthal magnetic field; i.e., the stabilized pinch. For the early period of the diffusion, a plane approximation is adequate. In this case, the equations may be solved in terms of a similarity variable proportional to original distance from the interface divided by the square root of time.

The results are shown in Fig. 5. The abscissa is the dimensionless similarity variable; $\xi = 0$ is the initial interface. The scale is such that $\xi = 1$ is about the skin depth which one would estimate using the conductivity deduced from the temperature which exists at $\xi = 1$. ϱ/ϱ_0 is the plasma density relative to its initial value; ϕ is the pitch angle of the magnetic field; and β is the ratio of material pressure to magnetic pressure. The significant feature of the results is that most of the energy liberated by uncrossing the fields is delivered to the plasma,[21] raising its pressure to about 0.43 of the initial magnetic pressure, regardless of the initial pressure. Hence, there exists a very efficient mechanism for creating very high temperature plasmas.

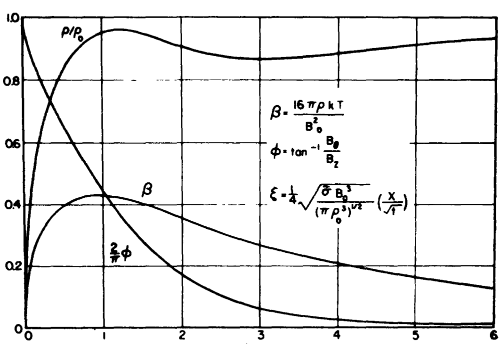

Figure 5. Solution of the dynamical equations for the plasma diffusion and field intermixing

Finally, we may inquire as to the eventual result of the diffusion process. If no external electric field is applied, the plasma will, of course, diffuse outward indefinitely until lost to the walls. On the other hand, an applied axial electric field is capable of causing a drift to balance out the collisional diffusion; and, in fact, detailed solutions have been found for a steady state of this type in collisional equilibrium away from the walls.[22] Unfortunately, this equilibrium is very unstable hydrodynamically.

To sum up, recent theoretical work has shown that the stabilized pinch is a self-heating device. On the other hand, the existence of surface instabilities will require a careful programming of magnetic fields. In addition, the large electric fields used in pinch formation may well lead to the formation of unstable plasma waves.

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Some Stable Plasma Equilibria in Combined Mirror-Cusp Fields

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The problem of equilibrium and stability of plasma confined in certain magnetic fields of combined mirror-cusp form is discussed. These fields have the properties that they are nowhere zero and everywhere increase toward the periphery. Attention is drawn to the importance of the existence of closed surfaces of constant $|B|$—the magnetic isobars. The conditions for plasma equilibrium are derived and interpreted; then by exploiting the existence of closed magnetic isobars certain low-β confined equilibria are constructed. These equilibria are shown to be stable according to the fluid (double adiabatic) energy principle and according to the small Larmor radius limit theory. A direct proof of stability against motions which preserve the magnetic moment is given. These equilibria have the property that there is no current along lines of force so that they are also immune to several drift instabilities.

I. INTRODUCTION

It is well known[1] that the adiabatic invariance of the magnetic moment of a charged particle provides a mechanism whereby plasma may be contained within magnetic mirrors; however mirror systems are usually hydromagnetically unstable.[2] It is generally believed that a hydromagnetically stable situation is provided by fields which increase away from the center,[3] as in the spindle cusp; in these systems the adiabatic invariance is usually destroyed by a weak field region near the center so that they are not genuine containment systems.

Recently there has been renewed interest[4] in magnetic field configurations which might provide both the inherent plasma stability attributed to fields whose strength increases towards the periphery, and the possibility of adiabatic containment.

One way of creating a field configuration of this type is by the addition of a multipole cusp field to the basic magnetic mirror (the stabilised mirror), as in the experiments of Ioffe.[4] Another method is by the insertion of a central, current-carrying conductor along the axis of a spindle cusp, thereby removing the weak field region which otherwise prevents adiabatic containment in the simple cusp.

General magnetic fields of the desired type can be identified by their two basic features, namely that there is a region in which (a) the field is nowhere zero, so that adiabatic containment is possible, and (b) the magnetic field strength $|B|$ "increases outwards."

By this second property of $|B|$ "increasing outwards" one means that there exists a point, or in some cases a closed curve, which is a local minimum of B^2. In the neighborhood of this point, or curve, the contours defined by B^2 = const form a set of closed, nested, surfaces and a surface of larger B^2 encloses those of smaller B^2. Since these surfaces *are* closed one can unambiguously refer to inside and outside; then one can say that the magnetic pressure is lower inside any given surface than outside it. It is in a region such as this that one hopes for stable plasma confinement and in this paper we prove that there exists at least one class of stable equilibria in these "nonvanishing outwardly increasing" fields.

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It should be emphasised that these surfaces of B^2 = const (which may be termed *magnetic isobars*) are *not* flux surfaces. A line of force will generally cut a magnetic isobar twice (or not at all) and the points of intersection could, for example, form the turning points of particles contained on that line by the mirror effect.

In Sec. II a brief description is given of an example of a "hybrid" mirror-cusp field configuration having the desired properties (a) and (b), while the main body of the paper, Secs. III–V, is concerned with finding low-β equilibria in such fields. Using a fluid description of the plasma the necessary and sufficient conditions for equilibrium are derived and are then interpreted in terms of individual particle motions. By exploiting the concept of closed magnetic isobars a class of confined low-β equilibria are then constructed which satisfy these equilibrium conditions. These have the property that $p_\perp$ and $p_\parallel$ are themselves constant over a magnetic isobar.

In Sec. VI the stability of this class of equilibria is discussed and they are shown to be stable against interchange instability (which is the only form of magnetohydrodynamic instability possible at low β) according to both the double adiabatic energy principle of Bernstein, *et al.*,[5] and the small-Larmor-radius-limit energy principle of Kruskal and Oberman.[6] Finally this stability is demonstrated in a more direct manner.

As the equilibria have the additional property that there is no current along the lines of force they should also be immune to several of the non-magnetohydrodynamic instabilities such as the drift instabilities.

II. MAGNETIC FIELD CONFIGURATION—AN EXAMPLE

As an example of the type of magnetic field under discussion we may consider the configuration employed by Ioffe.[4] Our object is merely to indicate some of the main features of this arrangement, particularly of the magnetic isobars.

Near the center of a mirror machine the field strength increases as one moves along the axis toward either mirror, but decreases as one moves radially away from the axis. A method of creating a field having the property that B^2 increases both axially and radially would therefore appear to be to superimpose on the mirror a second field which increases as one moves from the axis but which is constant along the axis. Such a field is the "multipole" field provided by $2l$ straight rods parallel to the axis of the machine, adjacent rods carrying current in opposite directions. Near the axis the multipole field is approximately

$$B_r = -\frac{lI^*}{R}\left(\frac{r}{R}\right)^{l-1}\cos l\theta, \qquad B_\theta = +\frac{lI^*}{R}\left(\frac{r}{R}\right)^{l-1}\sin l\theta, \tag{2.1}$$

where R is the distance of the rods from the axis and I^* is a measure of the current in each rod. (The relationship of I^* to the actual current I depends on the way that the current is distributed over the cross section of the rods and on the shape of this cross section; for thin rods $I^* = 2I$.) The original mirror field can be approximately represented by

$$B_z = B_0[1 - \alpha I_0(2\pi r/L)\cos(2\pi z/L)], \qquad B_r = -\alpha B_0 I_1(2\pi r/L)\sin(2\pi z/L), \tag{2.2}$$

where I_0 and I_1 are modified Bessel functions. The mirrors are situated at $z = \pm\frac{1}{2}L$ and the mirror ratio is

$$R_m = (1+\alpha)/(1-\alpha). \tag{2.3}$$

The formation of closed magnetic isobars of the required type can be illustrated easily when $l = 2$, for then near the center of the machine, $z = 0$, $r = 0$, the field strength is given by

$$B^2 = B_0^2(1-\alpha)^2 + 4\pi^2 B_0^2 \cdot\left\{\alpha(1-\alpha)\frac{z^2}{L^2} + \frac{r^2}{L^2}\left[\frac{I^{*2}L^2}{\pi^2 B_0^2 R^4} - \frac{\alpha(1-\alpha)}{2}\right]\right\}. \tag{2.4}$$

If the current in the multipole rods is small, so that

$$I^{*2} < (\pi^2 R^4/2L^2)\alpha(1-\alpha)B_0^2, \tag{2.5}$$

then the isobars form a family of hyperboloids. However, as the current in the multipole rods is increased so that

$$I^{*2} > (\pi^2 R^4/2L^2)\alpha(1-\alpha)B_0^2 \tag{2.6}$$

these magnetic isobars become closed (ellipsoidal) surfaces of the type we desire.

Before leaving this topic it is worth while noting that the situation is not so simple when $l > 2$. If $l > 2$ then sufficiently near the axis the multipole field is always too weak to compensate for the radial decrease in the basic mirror field. In this case closed magnetic isobars are still formed but instead of a single minimum at $r = 0$, $z = 0$, there are $2l$ minima situated off the axis.

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III. LOW-β EQUILIBRIA

We now consider the problem of plasma equilibrium in a magnetic field. For equilibrium the pressure tensor P must satisfy

$$\mathbf{j} \times \mathbf{B} = \nabla \cdot \mathsf{P} \qquad (3.1)$$

where $\mathbf{j}$ and $\mathbf{B}$ are connected by

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j}, \qquad (3.2)$$

$$\nabla \cdot \mathbf{B} = 0. \qquad (3.3)$$

A full solution to the problem of equilibrium would involve solving these equations subject to boundary conditions such as the given currents in the external conductors. However, apart from the impracticability of such a program, it is our present aim to derive general results independent of the detailed arrangement of conductors, and so applicable to all fields possessing properties (a) and (b) of Sec. I. We therefore seek low-β solutions (where β is the ratio of plasma pressure to magnetic pressure).

At zero β the magnetic field is the vacuum field due to external currents; this is easily calculated and will be considered as given. The first order perturbation in the field, due to plasma pressure, is given by:

$$\mathbf{j}_1 \times \mathbf{B}_0 = \nabla \cdot \mathsf{P}, \qquad (3.4)$$

$$\nabla \times \mathbf{B}_1 = 4\pi \mathbf{j}_1, \qquad (3.5)$$

$$\nabla \cdot \mathbf{B}_1 = 0, \qquad (3.6)$$

where $\mathbf{j}_1$ is the plasma current density, $\mathbf{B}_0$ the original vacuum field, and $\mathbf{B}_1$ the perturbation in this field due to the presence of plasma.

Now it might appear that these equilibrium equations should have solutions $\mathbf{j}_1$ and $\mathbf{B}_1$ for *any* given plasma pressure P and that there is, therefore, no problem. Indeed in axisymmetric configurations such as mirror or cusp this is true, but in general these equations will *not* possess a solution and our first task is to determine the conditions which P must satisfy in order that a solution should exist.

This is perhaps most easily done as follows: Eqs. (3.5) and (3.6) are simply the magnetostatic equations which are known to have a solution if $\mathbf{j}_1$ exists and $\nabla \cdot \mathbf{j}_1 = 0$. Our procedure therefore will be to solve Eq. (3.4) for $\mathbf{j}_1$ and then to examine under what conditions $\nabla \cdot \mathbf{j}_1 = 0$. (As we shall be concerned only with $\mathbf{j}_1$ and $\mathbf{B}_0$ we may henceforth suppress all subscripts, provided we remember that $\mathbf{B}$ *always* denotes a vacuum field.)

To illustrate the argument consider the case of scalar pressure when Eq. (3.4) reduces to

$$\mathbf{j} \times \mathbf{B} = \nabla p. \qquad (3.7)$$

The first necessary condition on p is clearly

$$\mathbf{B} \cdot \nabla p = 0 \quad \text{or} \quad \partial p/\partial s = 0, \qquad (3.8)$$

i.e., p is constant along a field line. Given that (3.8) is satisfied we can then solve (3.7) for $\mathbf{j}_\perp$ (the component of $\mathbf{j}$ perpendicular to $\mathbf{B}$),

$$\mathbf{j}_\perp = (-\nabla p \times \mathbf{B})/B^2, \qquad (3.9)$$

and therefore

$$\mathbf{j} = (-\nabla p \times \mathbf{B})/B^2 + \lambda \mathbf{B} \qquad (3.10)$$

where λ is an arbitrary scalar.

The requirement $\operatorname{div} \mathbf{j} = 0$ then gives

$$\mathbf{B} \cdot \nabla \lambda = \operatorname{div} (\nabla p \times \mathbf{B}/B^2), \qquad (3.11)$$

or

$$\mathbf{B} \cdot \nabla \lambda = -2 \nabla B \cdot (\nabla p \times \mathbf{B})/B^3. \qquad (3.12)$$

Equation (3.12) can be written

$$d\lambda/ds = -2 \nabla B \cdot (\nabla p \times \mathbf{B})/B^4, \qquad (3.13)$$

where s is measured along the line of force. A necessary condition for this equation to possess a unique single valued solution for λ is clearly

$$\oint \frac{\nabla B \cdot (\nabla p \times \mathbf{B})}{B^4} \, ds = 0, \qquad (3.14)$$

where the integral is taken along any closed line of force. Newcomb[7] has shown that this is also a sufficient condition.

In the case of scalar pressure, then, Eqs. (3.8) and (3.14) are the necessary and sufficient conditions which the pressure must satisfy if the plasma is to be in equilibrium. We now turn to the situation of immediate interest, namely when the pressure is anisotropic, and seek the analogous conditions on the pressure tensor.

Anisotropic Pressure

In a coordinate system with the principal axis along the magnetic field the pressure tensor can be written

$$\mathsf{P} = p_\perp \Pi + (p_\parallel - p_\perp)\mathbf{nn} \qquad (3.15)$$

where $\mathbf{n}$ is a unit vector in direction of $\mathbf{B}$ and Π is the unit tensor.

The momentum balance equation is now

$$\mathbf{j} \times \mathbf{B} = \nabla \cdot \mathsf{P}, \qquad (3.16)$$

[7] W. A. Newcomb, Phys. Fluids 2, 362 (1959).

and from the parallel component of this equation the first condition on $p_\perp$ and $p_\parallel$ is obtained,

$$\mathbf{n}\cdot\nabla p_\perp + \mathbf{n}\cdot\text{div}\,\{(p_\parallel - p_\perp)\mathbf{nn}\} = 0, \qquad (3.17)$$

or

$$\frac{\partial p_\parallel}{\partial s} + \frac{(p_\perp - p_\parallel)}{B}\frac{\partial B}{\partial s} = 0, \qquad (3.18)$$

where s is measured along the magnetic field. This condition specifies a relation between $p_\perp$ and $p_\parallel$ along a field line, replacing the simpler condition $\partial p/\partial s = 0$ of the scalar pressure theory. However, if (3.18) is satisfied then equation (3.16) can be solved for $\mathbf{j}_\perp$ as before,

$$\mathbf{j}_\perp = -\nabla p_\perp \times \mathbf{B}/B^2 + \mathbf{B}\times\text{div}\,[(p_\parallel - p_\perp)\mathbf{nn}]/B^2, \qquad (3.19)$$

and so

$$\nabla\cdot\mathbf{j}_\perp = 2\nabla p_\perp\cdot(\mathbf{B}\times\nabla B)/B^3 + \text{div}\,\{\mathbf{B}\times\text{div}\,[(p_\parallel - p_\perp)\mathbf{nn}]/B^2\}. \qquad (3.20)$$

It can be shown that because $\mathbf{B}$ is a vacuum magnetic field the last term can be transformed to give

$$\text{div}\,\{\mathbf{B}\times\text{div}\,[(p_\parallel - p_\perp)\mathbf{nn}]/B^2\} = \nabla(p_\parallel - p_\perp)\cdot(\mathbf{B}\times\nabla B)/B^3. \qquad (3.21)$$

Therefore we finally obtain

$$\nabla\cdot\mathbf{j}_\perp = \nabla(p_\perp + p_\parallel)\cdot(\mathbf{B}\times\nabla B)/B^3. \qquad (3.22)$$

Then, just as in the case of scalar pressure, the vanishing of $\nabla\cdot\mathbf{j}$ requires

$$\nabla\cdot\mathbf{j}_\parallel = \mathbf{B}\cdot\nabla\lambda = -\nabla\cdot\mathbf{j}_\perp \qquad (3.23)$$

so that

$$\mathbf{B}\cdot\nabla\lambda = -\nabla(p_\perp + p_\parallel)\cdot(\mathbf{B}\times\nabla B)/B^3. \qquad (3.24)$$

As before this can be written

$$d\lambda/ds = -\nabla(p_\perp + p_\parallel)\cdot(\mathbf{B}\times\nabla B)/B^4, \qquad (3.25)$$

and if the lines of force were closed this would lead to the condition

$$\oint \nabla(p_\perp + p_\parallel)\cdot\frac{(\mathbf{B}\times\nabla B)}{B^4}\,ds = 0. \qquad (3.26)$$

In the systems we are considering the lines of force are not closed within the plasma volume but leave the region of interest. In this case, provided the plasma is surrounded by a region in which no current flows, we must have

$$\int \nabla(p_\perp + p_\parallel)\cdot\frac{(\mathbf{B}\times\nabla B)}{B^4}\,ds = 0, \qquad (3.27)$$

where the integral is taken from the point where the line of force first enters the plasma to the point where it first leaves it. (If this condition were not satisfied λ would not be zero when the line of force left the plasma and there would be currents flowing in the plasma free region.) Furthermore, it is clear that if this condition (3.27) is satisfied, a unique λ can always be constructed from (3.25). The condition (3.27) is therefore both necessary and sufficient.

With anisotropic pressure, then, the necessary and sufficient conditions for equilibrium are (3.18) and (3.27). Before discussing some distributions satisfying these conditions we will first interpret these equilibrium constraints from the point of view of individual particle motions.

IV. PARTICLE MOTION

The first constraint (3.18) is simply the requirement that the particles be in equilibrium along each field line considered individually. This is entirely consistent with the basic idea of adiabatic mirror containment; for if the magnetic moment of a particle

$$\mu = V_\perp^2/2B \qquad (4.1)$$

is constant as it moves along a field line then

$$p_\perp \propto \int \rho(\mu, \epsilon)\frac{\mu B}{2}\,d\mu\,d\epsilon, \qquad (4.2)$$

$$p_\parallel \propto \int \rho(\mu, \epsilon)(\epsilon - \mu B)\,d\mu\,d\epsilon, \qquad (4.3)$$

where ρ is the local density of particles of specified magnetic moment μ and energy ϵ. This is proportional to (i) the number of such particles on the line $= f(\mu, \epsilon, L)$, (ii) to the density of lines $= B$, (iii) to the fraction of the time each particle spends near the point of interest

$$dt \propto dl/(\epsilon - \mu B)^{\frac{1}{2}}. \qquad (4.4)$$

Therefore, for particles contained by the mirror effect,

$$p_\perp = \int f(\mu, \epsilon, L)\frac{\mu B^2}{2(\epsilon - \mu B)^{\frac{1}{2}}}\,d\mu\,d\epsilon, \qquad (4.5)$$

$$p_\parallel = \int f(\mu, \epsilon, L)B(\epsilon - \mu B)^{\frac{1}{2}}\,d\mu\,d\epsilon. \qquad (4.6)$$

It can be verified by direct substitution that these expressions satisfy (3.18).

The second constraint (3.27) may be interpreted in terms of the guiding center drifts of the particles on a field line. As is well known,[2] the first order guiding center drift of a particle in an inhomogeneous magnetic field is

$$\mathbf{V}_D = \frac{mc}{e}\frac{(\mathbf{B}\times\nabla B)}{B^3}(\tfrac{1}{2}V_\perp^2 + V_\parallel^2), \qquad (4.7)$$

where $V_\perp$ is the velocity perpendicular to the field and $V_\parallel$ that along it.

The total current associated with this drift is then

$$\mathbf{j}_D = [(\mathbf{B}\times\nabla B)/B^3](p_\perp + p_\parallel) \qquad (4.8)$$

and the divergence of this expression is

$$\nabla\cdot\mathbf{j}_D = \nabla(p_\perp + p_\parallel)\cdot(\mathbf{B}\times\nabla B)/B^3, \qquad (4.9)$$

so that the second condition for equilibrium can be written

$$\int (\nabla\cdot\mathbf{j}_D)\frac{ds}{B} = 0. \qquad (4.10)$$

The meaning of this is made clear if we consider not the integral along a field line but the integral over an infinitesimal flux tube. This can be obtained by multiplying (4.10) by $B\,dA$ when we have

$$\int_{\text{Flux tube}} (\nabla\cdot\mathbf{j}_D)\,d\tau = 0, \qquad (4.11)$$

so that the condition found for the existence of a solution to the magnetostatic fluid equations is equivalent to the statement that the divergence of the current associated with the guiding center drifts should vanish when averaged over any flux tube. Of course, the current due to the guiding center drifts is *not* the same as the total current but the difference can be expressed as the curl of the magnetization per unit volume, whose divergence vanishes identically. The constraint might therefore equally well be applied to the total current or to the drift current.

V. A CLASS OF EQUILIBRIA

Now let us consider some particular solutions of the equilibrium constraints (3.18) and (3.27), appropriate to the type of magnetic field under discussion. It should first be noted that the second constraint (3.27) is not serious in systems of axial symmetry such as the mirror or the spindle cusp. For in these systems the symmetry ensures that ∇p, ∇B, and $\mathbf{B}$ are coplanar vectors (lying in the r, z, plane) so that the expression

$$\nabla(p_\perp + p_\parallel)\cdot\nabla B\times\mathbf{B} \qquad (5.1)$$

vanishes identically. Similarly in any cylindrically symmetric system ∇p and ∇B are both radial and (5.1) again vanishes.

In other field configurations the constraint (3.27) can be a severe restriction; for example, the condition (3.27) [or rather (3.26) which is then the appropriate form] can never be satisfied by any *confined* plasma distribution within a circular torus. For in such a configuration, symmetry ensures that the integral (3.26) can only vanish if the integrand vanishes. As $(\nabla B\times\mathbf{B})$ is in the direction parallel to the symmetry axis of the torus this means that p must be constant in this direction, thus the plasma is not confined. This, of course, is the well known lack of equilibrium in a simple toroidal field.

If we leave aside for the moment the question of whether it represents contained plasma or not, a restricted class of solutions to the equilibrium constraints can *always* be found by demanding that (5.1) should vanish. This is certainly achieved if $(p_\perp + p_\parallel)$ is a function only of B, then, since the "parallel" equilibrium equation (3.18) gives $p_\perp$ in terms of $p_\parallel$, this will make $p_\perp$ and $p_\parallel$ individually functions of B alone. Making $p_\perp$ and $p_\parallel$ functions of B alone means that the surfaces of constant B, the magnetic isobars, are also surfaces of constant $p_\perp$ and $p_\parallel$.

The significance of magnetic field configurations which possess closed magnetic isobars now becomes apparent. Equilibria in which $p_\perp$ and $p_\parallel$ are functions only of B exist in all field configurations, but only in those which possess closed magnetic isobars do these equilibria correspond to *confined* plasma configurations.

This class of low-β equilibria, which have

$$p_\perp = p_\perp(B), \qquad p_\parallel = p_\parallel(B), \qquad (5.2)$$

and, from (3.18),

$$Bp_\parallel' = p_\parallel - p_\perp \qquad (5.3)$$

where the prime denotes differentiation with respect to B, is one whose stability will be proved in the next section.

An example of this class of equilibrium distribution is

$$\begin{aligned} p_\parallel &= CB(B_0 - B)^n \\ p_\perp &= nCB^2(B_0 - B)^{n-1} \end{aligned} \quad \text{if } B < B_0, \qquad\qquad (5.4)$$
$$p_\perp = p_\parallel = 0 \quad \text{if } B > B_0,$$

where n, B_0 are arbitrary parameters. These equilibria correspond to plasma confined within the contour $B = B_0$ which, by the basic property of our fields, can be a closed contour.

Particle distribution functions corresponding to the equilibria (5.4) can also be written down in terms of the distribution in μ, ϵ space (see Sec. IV). A particle distribution function which leads to the pressure distributions (5.4) is

$$f(\mu, \epsilon) = (\mu B_0 - \epsilon)^{n-\frac{3}{2}} \cdot g(\mu), \qquad \epsilon < \mu B_0$$
$$f(\mu, \epsilon) = 0, \qquad \epsilon > \mu B_0 \tag{5.5}$$

where $g(\mu)$ is an arbitrary function of the magnetic moment.

VI. STABILITY OF THE SPECIAL EQUILIBRIA

To examine the stability of the equilibria described in the previous section let us first continue with a fluid description and consider the double adiabatic hydromagnetic energy principle derived by Bernstein *et al.*[5]

According to this, the stability of a plasma configuration with anisotropic pressure is determined by the sign of the minimum of the energy integral.

$$\delta W_D = \int d\tau \,\{|\mathbf{Q}|^2 - \mathbf{j}\cdot\mathbf{Q}\times\boldsymbol{\xi} + \tfrac{5}{3}p_\perp(\nabla\cdot\boldsymbol{\xi})^2$$
$$+ (\nabla\cdot\boldsymbol{\xi})(\boldsymbol{\xi}\cdot\nabla p_\perp) + \tfrac{1}{3}p_\perp[\nabla\cdot\boldsymbol{\xi} - 3q]^2$$
$$+ q\nabla\cdot[\boldsymbol{\xi}(p_\parallel - p_\perp)] - (p_\parallel - p_\perp)[\mathbf{n}\cdot(\mathbf{a}\cdot\nabla)\boldsymbol{\xi}$$
$$+ \mathbf{a}\cdot(\mathbf{n}\cdot\nabla)\boldsymbol{\xi} - 4q^2]\}, \tag{6.1}$$

where

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \qquad q = \mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\boldsymbol{\xi},$$
$$\mathbf{a} = (\mathbf{n}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{n}, \tag{6.2}$$

and $\boldsymbol{\xi}$ is an arbitrary displacement vector. $\delta W_{\min}$ should be positive for stability.

Examination of the energy integral shows that only the first term $|\mathbf{Q}^2|$ is independent of β so that at low β it must dominate (and so make δW positive) except for those displacements which themselves make $\mathbf{Q}$ zero. Physically these displacements are those which do not change the vacuum magnetic field—the so called interchange modes.

Hence, at sufficiently low β we can determine stability by examining δW for displacements which satisfy

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) = 0, \tag{6.3}$$

and for these displacements

$$q = \nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla B/B, \qquad \mathbf{a} = (\nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla B/B)\mathbf{n}.$$

With the aid of (6.3) and (6.4) the energy integral can be greatly simplified. In fact

$$\delta W_D = \int d\tau \,\{3p_\parallel d^2 + ds(5p_\parallel - p_\perp)$$
$$+ s^2(p_\perp + 2p_\parallel) + d(\boldsymbol{\xi}\cdot\nabla p_\parallel) + s[\boldsymbol{\xi}\cdot\nabla(p_\parallel - p_\perp)]\} \tag{6.5}$$

where, for brevity, we have written

$$\nabla\cdot\boldsymbol{\xi} \equiv d, \qquad \boldsymbol{\xi}\cdot\nabla B/B \equiv s.$$

So far this is quite general. For the equilibria found in Sec. V, namely those which have the properties

$$p_\perp = p_\perp(B), \qquad p_\parallel = p_\parallel(B), \qquad Bp'_\parallel = p_\parallel - p_\perp, \tag{6.6}$$

δW_D reduces to

$$\delta W_D = \int d\tau \left\{\frac{1}{3p_\parallel}[3p_\parallel(d + s) - p_\perp s]^2 + s^2\left[2p_\perp - \frac{p_\perp^2}{3p_\parallel} - Bp'_\perp\right]\right\}. \tag{6.7}$$

The first term is clearly non-negative so a sufficient criterion for stability according to the double adiabatic principle is

$$2p_\perp - (p_\perp^2/3p_\parallel) - Bp'_\perp > 0. \tag{6.8}$$

Some explicit examples of equilibria were given by equations (5.4). For these examples

$$Bp'_\perp = 2p_\perp - [(n - 1/n]p_\perp^2/p_\parallel \tag{6.9}$$

and a sufficient stability condition is $n > \frac{3}{2}$. [Note that this is also the condition for $f(\mu, \epsilon)$ in Eq. (5.5) to be continuous at $\epsilon = \mu B_0$.]

There is, however, one reservation to be made about the argument above. The last term in the energy integral contains the expression $p_\perp^2/p_\parallel$ and for some of the equilibria of the form (5.4) this quantity tends to infinity at the plasma boundary. This will make possible, even at low β, some instabilities in which the magnetic field *is* perturbed. These are the "mirror" instabilities. As the plasma density falls to zero at the surface it is not clear whether this particular instability is to be taken seriously, but in any case it can be avoided by demanding that $p_\perp^2/p_\parallel$ be finite at the surface. This requirement is satisfied by the equilibria given in (5.4) if $n > 2$.

The Small Larmor Radius Theory

The double adiabatic energy principle is open to two objections; firstly, that it is based on the assumption that in the plasma motion there is no heat flow along the lines of force, and secondly, that although a component of the displacement ξ along the lines of force is formally allowed, it is hard to see what is the real significance of this parallel displacement (since, in collisionless plasma, motion arises from $\mathbf{E} \times \mathbf{B}$ drifts).

An energy principle which is sufficient, though not necessary, for stability and which overcomes these objections was given by Kruskal and Oberman.[6] This is based on the use of the Boltzmann equation in the limit of small Larmor radius. In this case the appropriate energy integral can be written

$$\delta W_{k0} = \delta W_D - \int d\tau \, \{2p_\perp q(\nabla\cdot\xi) + (3p_\parallel - 2p_\perp)q^2\} + I, \quad (6.10)$$

where

$$I = \int d\tau \left\{ \sum m_i \iint \frac{B}{V_\parallel} d\mu \, d\epsilon \cdot \left[\mu^2 B^2 \left(\frac{\partial f_0}{\partial \epsilon}\right)(\nabla\cdot\xi - q)^2 - \frac{f^{*2}}{\partial f_0/\partial\epsilon} \right] \right\} \quad (6.11)$$

and

$$\tfrac{1}{2} V_\parallel^2 = \epsilon - \mu B. \quad (6.12)$$

In these expressions ϵ and μ are again the energy and magnetic moment as in Sec. V, and f^* is the perturbation in the particle distribution function. The quantity $f_0(\mu, \epsilon, L)$ is the unperturbed particle distribution, and in their derivation of the energy principle Kruskal and Oberman require that

$$\partial f_0/\partial\epsilon < 0. \quad (6.13)$$

The minimization of δW_{k0} has to be carried out over ξ and also over f^* subject to certain constraints. The minimization over f^* is carried out in the Kruskal and Oberman paper but we will have no need of this in the present discussion.

It can be shown that the minimum of δW_{k0} is independent of $\xi_\parallel$ as it should be, so that $\xi_\parallel$ can be taken to be zero.

As before, at sufficiently low β we need only consider displacements which satisfy

$$\nabla \times (\xi \times \mathbf{B}) = 0 \quad (6.14)$$

so that Eqs. (6.4) are again valid. However as ξ is now perpendicular to $\mathbf{B}$ a further simplification can also be obtained. For (6.14) implies that

$$\xi \times \mathbf{B} = \nabla\phi, \quad (6.15)$$

and so ξ can now be written

$$\xi \equiv \xi_\perp = \mathbf{B} \times \nabla\phi/B^2, \quad (6.16)$$

whence

$$\nabla\cdot\xi = -2\xi\cdot\nabla B/B. \quad (6.17)$$

With the aid of Eqs. (6.4) and (6.17) the energy integral may be reduced to

$$\delta W_{k0} = \int d\tau \, \{(2p_\perp - Bp'_\perp)s^2\} + \int d\tau \sum m_i \iint \frac{B}{V_\parallel} d\mu \, d\epsilon \cdot \left[\mu^2 B^2 \left(\frac{\partial f_0}{\partial\epsilon}\right) s^2 - \frac{f^{*2}}{\partial f_0/\partial\epsilon} \right]. \quad (6.18)$$

This can be further simplified, for

$$p_\perp = \sum m_i \iint \frac{\mu B^2}{V_\parallel} f_0 \, d\mu \, d\epsilon, \quad (6.19)$$

and since

$$1/V_\parallel = \partial V_\parallel/\partial\epsilon, \quad (6.20)$$

a partial integration leads to

$$p_\perp = -\sum m_i \iint \mu B^2 V_\parallel \left(\frac{\partial f_0}{\partial\epsilon}\right) d\epsilon \, d\mu. \quad (6.21)$$

Then if $p \equiv p_\perp(B)$, differentiation with respect to B gives

$$Bp'_\perp = 2p_\perp + \sum m_i \iint \frac{\mu^2 B^3}{V_\parallel} \left(\frac{\partial f_0}{d\epsilon}\right) d\epsilon \, d\mu \quad (6.22)$$

and using this result the energy integral is finally reduced to

$$\delta W_{k0} = -\int d\tau \sum m_i \iint \frac{B}{V_\parallel} d\mu \, d\epsilon \left\{ \frac{f^{*2}}{\partial f_0/\partial\epsilon} \right\}, \quad (6.23)$$

which is certainly positive if $\partial f_0/\partial\epsilon < 0$, a condition which is in any case required for the present energy principle to be valid.

According to the small Larmor radius theory of Kruskal and Oberman, then, equilibria of the class (6.6) are stable if their corresponding particle distributions satisfy

$$\partial f_0/\partial\epsilon < 0. \quad (6.24)$$

Now, the specific examples (5.4) correspond to the particle distributions (5.5) and so are stable if

$$(\partial/\partial\epsilon)(\mu B_0 - \epsilon)^{n-\frac{3}{2}} < 0, \quad (6.25)$$

that is if $n > \frac{3}{2}$. In this case, therefore, the two energy principles lead to the same criterion.

VII. DIRECT PROOF OF STABILITY

The simplicity of the form of the final expression for δW_{k0} suggests that a more direct demonstration of the stability of our equilibria should be possible which did not make use of the full Kruskal–Oberman theory. Such a proof of stability can be developed by extension of the argument given by Newcomb[8] in discussing stability of infinite Maxwellian plasma.

Let us consider a general particle motion in which the magnetic moment of a particle is invariant, (as in small Larmor radius theory), then a general constant of the motion constructed from individual particle constants is

$$S = \int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau\, G(f, \mu). \tag{7.1}$$

Now consider a distribution function $f = f_0 + \delta f$, where f_0 is the initial equilibrium distribution whose stability we want to discuss. Then we can write

$$\delta S = 0 = \int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau \left\{ G'(f_0, \mu)\, \delta f + G''(f_0, \mu) \frac{(\delta f)^2}{2} + \cdots \right\}, \tag{7.2}$$

where

$$G'(f, \mu) \equiv \partial G/\partial f.$$

Now the equilibria we are considering have the property that $p_\perp$ and $p_\parallel$ are functions of B only and satisfy the parallel equilibrium equation. Such equilibria correspond to particle distribution functions which depend only on μ and ϵ [i.e., $f_0(\epsilon, \mu, L)$ is independent of the particular flux line considered]. For these equilibria, therefore, the function G can be chosen so that

$$G'(f_0, \mu) \equiv \epsilon \tag{7.3}$$

(at least if $\partial f_0/\partial \epsilon$ is monotonic) and with this choice for G Eq. (7.2) becomes

$$\int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau\, (\epsilon\, \delta f) = -\int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau \frac{(\delta f)^2}{2(\partial f_0/\partial \epsilon)} + \cdots, \tag{7.4}$$

which may be written

$$\delta K = -\int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau \frac{(\delta f)^2}{2(\partial f_0/\partial \epsilon)} + \cdots \tag{7.5}$$

where K is the total kinetic energy of the particles,

$$K = \int \frac{B}{V_\parallel} d\mu\, d\epsilon\, d\tau\, (\epsilon f).$$

If now $\partial f_0/\partial \epsilon < 0$, it is clear that to second order in δf, $\delta K > 0$ so that any change δf in f around f_0 will *increase* the kinetic energy. Furthermore if the equilibrium has no electric fields and is of such low β that the magnetic field is a vacuum field, then any perturbations can also only increase the field energies. As the total energy is constant it is clear that δf cannot grow indefinitely and in particular cannot grow exponentially. Therefore the system is stable.

Thus it has been shown that any low-β equilibrium with f_0 a function only of μ and ϵ is stable against all perturbations in which the magnetic moment is an invariant. This is certainly sufficient to demonstrate stability against hydromagnetic motions.

VIII. CONCLUSIONS

Attention has been drawn to the importance of the existence of closed magnetic isobars in certain hybrid mirror–cusp magnetic fields. The existence of these closed isobars enables one to construct a class of confined plasma distributions, those with $p_\perp$ and $p_\parallel$ functions of B alone, which satisfy the conditions for equilibrium. These equilibria are stable against interchanges according to both the double-adiabatic energy principle and the more complete small-Larmor-radius theory. A direct proof of stability against all motions in which the magnetic moment of a particle is an invariant has also been given.

It is easily shown that these equilibria have the property that $j_\parallel = 0$, which ensures that they are also stable against several forms of "drift" instability; the large amount of "shear" and the high curvature of some of these hybrid fields may also inhibit some other micro-instabilities. One concludes, therefore, that these nonvanishing outwardly increasing fields do indeed offer the possibility of stable plasma confinement.

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[8] See I. B. Bernstein, Phys. Rev. 109, 10 (1958); also M. D. Kruskal and C. R. Oberman, Phys. Fluids 1, 275 (1958).